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Applications of maximal topologies

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Abstract

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We construct some unusual spaces by considering maximal members of suitable families of topologies. For example, we construct a countable regular crowded space no point of which is a limit point of two disjoint sets. An application to ω^* is that there is a separable space which is a continuous image of ω^* under a $\leq$two-to-one map. We also show that for each $k \in [2, \omega)$, there is a k-irresolvable space.

Keywords: Perfectly disconnected space, $\leq$two-to-one map.

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Our main goal is to prove that the space ω^*, which is a compact space which is very far from being separable, can be mapped onto a separable space by a function each of whose fibers has at most two elements. We do this in Section 4 by showing that if there is a countable, crowded, perfectly disconnected, regular space, then the Čech-Stone compactification of this space is the image of ω^* under the required type of map. The problem is therefore to find a countable, crowded, perfectly disconnected, regular space. This is what we do in the first three sections. In the last section, using some of the earlier results of the paper we show that for each integer k there is a k-irresolvable countable regular space.

1. Auxiliary results

Definition 1.1. Call a space X *maximal* if τX is maximal in the collection of all crowded topologies on X. If $\mathcal{T}$ is a separation axiom call X *maximal* $\mathcal{T}$ if τX is maximal in the poset of all crowded topologies with $\mathcal{T}$ on the underlying set of X.

These concepts are due to Hewitt, [4].

Theorem 1.2. (a) *There exists a (countable) maximal regular space.*

(b) *There exists a Hausdorff maximal space.*

(c) *Maximal Hausdorff spaces are maximal.*

(d) *A space is maximal iff it is maximal T_1.*

Proof. (d) only if: If X is a maximal space, then

$$\mathcal{T} = \{U\setminus F\colon U \in \tau X \text{ and } F \text{ is a finite subset of } X\}$$

is a crowded T_1-topology on X with $\mathcal{T} \supseteq \tau X$; hence $\tau X = \mathcal{T}$.

(c) and (d) if: For $i \in \{1, 2\}$, if $\mathcal{S}$ and $\mathcal{T}$ are topologies with $\mathcal{S} \subseteq \mathcal{T}$ and if $\mathcal{S}$ is T_i, then $\mathcal{T}$ is T_i.

(a) and (b): There is a crowded regular topology $\mathcal{T}$ on ω. For $i \in \{2, 3\}$ let $\mathbb{P}_i$ be the poset, under inclusion, of all crowded T_i-topologies $\mathcal{S}$ on ω with $\mathcal{S} \supseteq \mathcal{T}$. To see $\mathbb{P}_i$ has a maximal member note that if $\mathbb{C}$ is a chain in $\mathbb{P}_i$ then $\bigcup\mathbb{C}$ is a collection closed under finite intersection every member of which is infinite, hence $\bigvee\mathbb{C}$, the topology $\bigcup\mathbb{C}$ is a base for, is crowded. Clearly, $\bigvee\mathbb{C}$ is T_2 since $\bigvee\mathbb{C} \supseteq \mathcal{T}$. To see that $\bigvee\mathbb{C}$ is regular if $i = 3$, consider any $\mathcal{C} \in \mathbb{C}$, any $C \in \mathcal{C}$, and any $x \in C$. There are U, V belonging to $\mathcal{C}$, hence to $\bigvee\mathbb{C}$, such that $x \in U \subseteq V^c \subseteq C$. Therefore $\bigvee\mathbb{C} \in \mathbb{P}_i$. Hence $\mathbb{P}_i$ has a maximal member by Zorn's lemma. □

Definition 1.3. A space X is *perfectly disconnected* if no point of X is a limit point of disjoint subsets of X, or equivalently if $p \in \overline{A\setminus\{p\}}$ implies that $p \in (A \cup \{p\})^c$.

We will see later that the crowded perfectly disconnected spaces are precisely the maximal spaces. Therefore, in order to find a regular perfectly disconnected crowded space, we need to find a regular maximal space. (Note that a regular maximal space is not the same thing as a maximal regular space.) Unfortunately, it is not true that if $\mathcal{S}$ and $\mathcal{T}$ are topologies with $\mathcal{S} \subseteq \mathcal{T}$ and if $\mathcal{S}$ is T_3, then $\mathcal{T}$ is T_3. Indeed, in Example 1.9, we will see that there is a space that is maximal regular but not maximal. It will take some effort to overcome this and construct a maximal regular space anyway.

A useful property of the spaces introduced is given by:

Definition and fact 1.4. A space X is called *ultradisconnected* if it is crowded and if

(a) every two disjoint crowded subsets of X have disjoint closures; or, equivalently, if

(b) a nonempty proper subset A of X is clopen iff both A and A^c are crowded.

Proof. The implication (a) ⇒ (b) is clear.

(b) ⇒ (a): Let A and B be disjoint crowded subsets of X. First notice that both $\overline{A \cup B}$ and $\overline{A \cup B}^c$ are crowded. Hence, $\overline{A \cup B}$ is clopen.

Define

$$A^* = \bar{A}\backslash B \quad \text{and} \quad B^* = \bar{B}\backslash A^*.$$

Then clearly

$$A^* \cap B^* = \emptyset \quad \text{and} \quad A \subseteq A^* \subseteq \bar{A} \quad \text{and}$$

$$B \subseteq B^* \subseteq \bar{B} \quad \text{and} \quad A^* \cup B^* = \overline{A \cup B}.$$

Clearly this implies $A^{*c} = B^* \cup \overline{A \cup B}$, and also that A^* and B^* are crowded. As $\overline{A \cup B}^c$, being open, is also crowded, it follows that A^* is clopen, and therefore, $\bar{A} \cap \bar{B} = \emptyset$ since $A \subseteq A^*$ and $B \cap A^* = \emptyset$. □

Fact 1.5. *Let X be ultradisconnected. Then*

(a) *Every closed crowded subspace of X is open.*

(b) *X is extremally disconnected.*

(c) *Every crowded subspace of X is ultradisconnected.*

Proof. (a): By definition ultradisconnected spaces are crowded, so closed subsets have crowded complements.

(b): Clear from (a).

(c): Trivial. □

Fact 1.6. *If $i \in \{1, 2, 3\}$ then maximal T_i-spaces are ultradisconnected.*

Proof. Let $A \subseteq X$ be such that both A and A^c are crowded. Then the topological sum $A + A^c$ of A and A^c is crowded and T_i, and clearly $\tau(A + A^c) \supseteq \tau X$. Hence $\tau(A + A^c) = \tau X$, and therefore A is clopen in X since it is clopen in $A + A^c$. □

Corollary 1.7. *Maximal regular spaces are extremally disconnected and hence zero-dimensional.*

Theorem 1.8. *A space is maximal regular iff it is regular and ultradisconnected.*

Proof. *Necessity* follows from Fact 1.6.

Sufficiency: Let $\mathcal{S} \supseteq \tau X$ be a regular crowded topology. Consider any $S \in \mathcal{S}$. To prove $S \in \tau X$ consider any $x \in S$. Since $\langle X, \mathcal{S}\rangle$ is regular there is $U \in \mathcal{S}$ with $x \in U$ and $\mathrm{cl}_{\mathcal{S}} U \subseteq S$. As $\mathcal{S}$ is crowded, both $\mathrm{cl}_{\mathcal{S}} U$ and $X \backslash \mathrm{cl}_{\mathcal{S}} U$ are crowded in $\langle X, \mathcal{S}\rangle$, hence in X since $\mathcal{S} \supseteq \tau X$. Hence $\mathrm{cl}_{\mathcal{S}} U \in \tau X$ since X is ultradisconnected. This shows that x is in the interior with respect to τX of S. □

Example 1.9. There is a space which is maximal regular but not maximal.

Proof. Let M be a maximal regular space, which exists by Theorem 1.2(a). Choose a fixed element q of M. Let p be a free ultrafilter on ω. We will define a topology

$\mathcal{T}$ on the set $X = (\omega \times M) \cup \{p\}$. For $n \in \omega$ and U open in M, the set $\{n\} \times U$ is open. If $p \in A \subseteq X$, then A is a neighborhood of p if $\{n \in \omega$: $A \cap (\{n\} \times M)$ is a neighborhood of $(n, q)\} \in p$. It is easy to check that this defines a regular topology on X. The topology $\mathcal{T}$ is not maximal because a strictly stronger crowded topology arises by declaring the set $\omega \times \{q\}$ to be closed. We still must show that $\langle X, \mathcal{T}\rangle$ is maximal regular, or equivalently by Theorem 1.8 that it is ultradisconnected. Suppose A is a subset of X such that A and $X \backslash A$ are crowded. By Definition and fact 1.4, we are done if we can show that A is clopen. Assume without loss of generality that $p \in A$. Then applying Definition and fact 1.4 to each set $\{n\} \times M$ gives that $X \backslash A$ is open. Similarly, A is a neighborhood of each of its elements other than p. So we need to check that A is a neighborhood of p. But if not, then the set $\{n \in \omega$: $A \cap (\{n\} \times M)$ is not a neighborhood of $(n, q)\} \in p$, so $\{n \in \omega$: $(X \backslash A) \cap (\{n\} \times M)$ is a neighborhood of $(n, q)\} \in p$ (again by Definition and fact 1.4). But then p is isolated in A since $\{p\} \cup (X \backslash A)$ is a neighborhood of p, contradicting the fact that A is crowded. □

Definition 1.10. A space is *irresolvable* if it is crowded and no dense subset has dense complement. A space is *hereditarily irresolvable* if it is crowded and every crowded subset is irresolvable. A space is *open-hereditarily irresolvable* if it is crowded and every open subset is irresolvable.

Here "hereditarily irresolvable" is due to Hewitt, [4], who essentially pointed out the following:

Fact 1.11. *Ultradisconnected spaces are hereditarily irresolvable.*

Note that every hereditarily irresolvable space is open-hereditarily irresolvable; the converse is false:

Example 1.12. There is an open-hereditarily irresolvable space that is not hereditarily resolvable.

Proof. By Theorems 1.2(a), 1.8, and Fact 1.11 there is an irresolvable space, S say. Let A be a space with a dense set I of isolated points, such that $A \backslash I$ is resolvable, e.g. the Alexandroff double of $\mathbb{Q}$, the rationals, or the subspace $\mathbb{Q} \times \{0\} \cup \{\langle k/n, 1/n\rangle$: $k \in \mathbb{Z}$ and $n \in \mathbb{N}\}$ of $\mathbb{Q}^2$, and let X be the space obtained from A by replacing every point in I by a copy of S. Formally, $X = (A \backslash I) \cup (I \times S)$ and a subset U of X is open if and only if $(\forall x \in (A \backslash I) \cap U)(\exists$ open $V \subseteq A)[x \in V$ and $V \cap (A \backslash I) \subseteq U$ and $(V \cap I) \times S \subseteq U]$, and $U \cap (I \times S)$ is open in $I \times S$ (with the product topology). □

We introduce open-hereditarily irresolvable because we have a characterization.

Fact 1.13. *For a crowded space X the following are equivalent*:

(a) *X is open-hereditarily irresolvable*; *and*

(b) $(\forall A \subseteq X)$ [$A^\circ = \emptyset \Rightarrow A$ *is nowhere dense*].

(Note that in (b) trivially $\Leftarrow$ holds.)

Proof. (a) $\Rightarrow$ (b): Consider any $A \subseteq X$ with $A^\circ = \emptyset$ and any nonempty open U in X. As $A^\circ = \emptyset$, $U \setminus A$ is dense in U. As U is irresolvable, it follows that A is not dense in U.

(b) $\Rightarrow$ (a): Let Y be any nonempty open subspace of X, and consider any dense subset A of Y. Then A is not nowhere dense, hence $A^\circ \neq \emptyset$. As $A \subseteq Y$ it follows that $Y \setminus A$ is not dense in Y. Hence, Y is irresolvable. □

Let us consider a class of spaces in which nowhere dense subsets are very small:

Definition and fact 1.14. We call a space X a *nodec* space if it satisfies one of the following equivalent properties:

(a) every nowhere dense subset is closed;
(b) every nowhere dense subset is closed discrete;
(c) every subset containing a dense open subset is open.

Proof. (a) $\Rightarrow$ (b): A set is closed discrete iff each of its subsets is closed.

All other implications are trivial. □

Fact 1.15. *An ultradisconnected space is nodec iff every relatively discrete subset is closed.*

Proof. *Necessity* is clear. To prove *sufficiency*, let A be any nowhere dense subset. We claim A has no crowded subspaces. Indeed, if $B \subseteq A$ then B^c is dense in X, hence is crowded, so B is not crowded. This proves that the set I of isolated points of A is dense in A. But I is closed by hypothesis. Therefore A is closed discrete.

(This proof actually shows that it suffices to know that no relatively discrete subset has only one cluster point. However, I do not know how to exploit this.) □

2. A characterization of perfectly disconnected crowded spaces

In the following, X^d denotes the derived set of X, i.e., the set of nonisolated points of X.

Theorem 2.1. *For a crowded space X the following are equivalent*:

(a) *X is perfectly disconnected*;
(b) $(\forall x \in X^d)[\{A \subseteq X : x \in \overline{A \setminus \{x\}}\}$ is an ultrafilter];
(c) $(\forall A \subseteq X)(\forall x \in A)[x \in \overline{A \setminus \{x\}} \Rightarrow x \in A^\circ]$.

Proof. (a) $\Rightarrow$ (b): Fix any $x \in X^d$, and let

$$\mathcal{I} = \{A \subseteq X : x \notin \overline{A \setminus \{x\}}\}.$$

We must show that $\mathscr{I}$ is a prime ideal. Clearly $X \notin \mathscr{I}$ since x is not isolated and clearly $\mathscr{I}$ is closed under finite unions. Also, $\mathscr{I}$ is prime, i.e., $(\forall A \subseteq X)[A \in \mathscr{I}$ or $X \backslash A \in \mathscr{I}]$ because X is perfectly disconnected.

(b) $\Rightarrow$ (a): Trivial.

(a) $\Rightarrow$ (c): Consider any $A \subseteq X$ and $x \in A \cap X^d$. If $x \in A^\circ$ then $x \in \overline{A \backslash \{x\}}$ since $x \in X^d$. If $x \in \overline{A \backslash \{x\}}$ then $x \notin \overline{(X \backslash A) \backslash \{x\}}$ since X is perfectly disconnected; hence $x \notin \overline{X \backslash A}$ since $x \in A$, so $x \in A^\circ$.

(c) $\Rightarrow$ (a): Suppose $A \subseteq X$ and $x \in \overline{A \backslash \{x\}}$. Then $x \in \overline{(A \cup \{x\}) \backslash \{x\}}$ so applying (c) to $A \cup \{x\}$ gives $x \in (A \cup \{x\})^\circ$. □

We now come to the main result of this section, a characterization of perfect disconnectedness among crowded spaces. In Section 3 we will recognize that a certain crowded space we construct is perfectly disconnected because it satisfies (d).

Theorem 2.2. *For a crowded space X the following are equivalent*:

(a) *X is perfectly disconnected*;

(b) *a subset of X is open iff it is crowded*;

(c) *X is maximal*;

(d) *X is ultraconnected and nodec*;

(e) *X is extremally disconnected, open-hereditarily irresolvable, and nodec.*

Proof. (a) $\Rightarrow$ (b): Let $A \subseteq X$. If A is crowded then it is open because of (a) $\Rightarrow$ (c) of Theorem 2.1. And if A is open then it is crowded since X is crowded.

(b) $\Rightarrow$ (c): Trivial.

(c) $\Rightarrow$ (d): X is ultradisconnected because of Fact 1.6 and Theorem 1.2(d). To prove X is nodec consider any dense set D. Clearly

$$\mathscr{S} = \{U \cup (V \cap D): U, V \in \tau X\}$$

is a topology on X with $\mathscr{S} \supseteq \tau X$. We prove that $\langle X, \mathscr{S} \rangle$ is crowded: Consider any $x \in X$. Since X is crowded, if $U \in \tau X$ then $U \neq \{x\}$. Next consider any $V \in \tau X$. Then $V \backslash \{x\}$ is nonempty, since X is crowded, and $V \backslash \{x\}$ is open since X is T_1, by Theorem 1.2(d). Hence $(V \backslash \{x\}) \cap D \neq \emptyset$ since D is dense in X. This proves that $V \cap D \neq \{x\}$.

(d) $\Rightarrow$ (e): Ultradisconnected spaces are extremally disconnected and open-hereditarily irresolvable by Facts 1.5 and 1.11.

(e) $\Rightarrow$ (a): We will prove that X satisfies condition (c) of Theorem 2.1. So consider any $A \subseteq X$ and any $x \in A$ such that $x \in \overline{A \backslash \{x\}}$.

We first show that $x \in \overline{A^\circ}$. Since clearly $(A \backslash A^\circ)^\circ = \emptyset$ and since X is open-hereditarily irresolvable, it follows from Fact 1.13 that $A \backslash A^\circ$ is nowhere dense, hence it is closed discrete since X is nodec. Therefore, $x \notin \overline{(A \backslash A^\circ) \backslash \{x\}}$ which implies $x \in \overline{A^\circ \backslash \{x\}}$.

As X is extremally disconnected, $\overline{A^\circ}$ is open. Also, $\overline{A^\circ} \backslash A^\circ$, the boundary of A°, is nowhere dense, hence is closed discrete. Therefore, $(\overline{A^\circ} \backslash A^\circ) \backslash \{x\}$ is closed. As $\overline{A^\circ}$ is open it follows that

$$A^\circ \cup \{x\} = \overline{A^\circ} \setminus ((\overline{A^\circ} \setminus A^\circ) \setminus \{x\})$$

is open. Since $x \in A$, this implies $x \in A^\circ$. □

3. Better subspaces

Here we investigate ways to get better subspaces inside the type of spaces we are looking at.

Fact 3.1. *Every irresolvable space has a hereditarily irresolvable (nonempty) open subspace.*

Proof. Let X be irresolvable, and let $\mathcal{U}$ be a maximal pairwise disjoint collection of crowded resolvable subspaces. A moment's reflection shows that $\bigcup \mathcal{U}$ is resolvable, hence that $\mathrm{cl} \bigcup \mathcal{U}$ is resolvable. Hence $Y = X \setminus \mathrm{cl} \bigcup \mathcal{U}$ is a nonempty open subspace of X. If S is any nonempty crowded subspace of Y then $\mathcal{U} \cup \{S\}$ is a pairwise disjoint collection of subsets of X which has $\mathcal{U}$ as a proper subcollection, hence, by maximality of $\mathcal{U}$, S is not resolvable. □

We now come to the main idea of the construction of perfectly disconnected spaces.

Lemma 3.2. *If X is a countable regular open-hereditarily irresolvable space, then*

$$A_X = \{x \in X : (\exists D \subseteq X)[D \textit{ is relatively discrete and } x \in \bar{D} \setminus D]\}$$

is nowhere dense.

Proof. Because of Fact 1.13 we prove A_X is nowhere dense if we prove $A_X^\circ = \emptyset$. Since

$$(\forall x \in A_X^\circ)(\exists D \subseteq A_X^\circ)[D \text{ is relatively discrete and } x \in (\mathrm{cl}_{A_X} D) \setminus D]$$

it suffices to show that $A_X \neq X$; for if $A_X \neq X$ for any choice of X, then in particular, $A_U \neq U$ when U is a nonempty clopen (in X) subset of A_X. (Notice that X being countable and regular, is zero-dimensional.) Let A_X be denoted A, and we show $A \neq X$.

So suppose that $A = X$. Let $s : \omega \to X$ be a surjection. We claim that there is a pairwise disjoint sequence $\langle D_n : n \in \omega \rangle$ of relatively discrete subsets of X such that

(1) $(\forall k < n \in \omega)[\underline{D_k} \subseteq \overline{D_n}]$ and

(2) $(\forall n \in \omega)[s_n \in \overline{D_n}]$.

However, that is impossible, for then clearly both $\bigcup_{n \in \omega} D_{2n}$ and $\bigcup_{n \in \omega} D_{2n+1}$ are dense in X.

We construct the D_n as follows: $D_0 = \{s_0\}$. Next, let $n \in \omega$ and suppose D_k is known for $k \leq n$. Let $Y = X \setminus (\overline{D_n} \setminus D_n)$. Since D_n is countable and since X is regular, there is a pairwise disjoint indexed open collection $\langle U_x : x \in D_n \rangle$ such that

(3) $(\forall x \in D_n)[x \in U_x \subseteq Y]$;

since D_n is relatively discrete, so that $\overline{D_n} \setminus D_n$ is closed, we see from (1) that we may assume

(4) $(\forall x \in D_n)[U_x \cap \bigcup_{k<n} D_k = \emptyset]$.

For each $x \in D_n$ we have $x \in U_x \subseteq Y \subseteq A$; hence we can choose a relatively discrete D_x with $x \in \operatorname{cl} D_x \setminus D_x$ and $D_x \subseteq U_x$. Let $T = \bigcup\{D_x : x \in D_n\}$, and if $s_{n+1} \in \bar{T}$ let $D_{n+1} = T$; otherwise, let $D_{n+1} = T \cup \{s_{n+1}\}$.

It is easy to verify that this D_{n+1} is as required: Obviously $D_n \subseteq \overline{D_{n+1}}$, so $(\forall k \leq n)$ $[D_k \subseteq \overline{D_{n+1}}]$ because of (1). Also clearly $D_{n+1} \cap \bigcup_{k<n} D_k \subseteq D_{n+1} \cap \bigcup\{U_x : x \in D_n\} = \emptyset$ because of (4), and $D_{n+1} \cap D_n = \emptyset$ because for each $x \in D_n$ we have $D_x \cap D_n \subseteq D_x \cap U_x \cap D_n = D_x \cap \{x\} = \emptyset$. Since $D_{n+1} = T$ except when $s_{n+1} \notin \bar{T}$, and since $(\forall k \leq n)$ $[D_k \subseteq \bar{T}]$, this shows $(\forall k \leq n)[D_k \cap D_{n+1} = \emptyset]$. Finally T is relatively discrete since $\langle U_x : x \in D_n \rangle$ is a pairwise disjoint indexed open collection and since for each $x \in X$ the set D_x is relatively discrete with $D_x \subseteq U_x$. As $D_{n+1} = T$ except when $s_{n+1} \notin \bar{T}$, this shows that D_{n+1} is relatively discrete. □

Example 3.3. There is a countable, perfectly disconnected, regular, crowded space.

Proof. We will find a countable, regular, crowded space which satisfies condition (d) of Theorem 2.2. By Theorem 1.2(a) there is a countable maximal regular space X, which, by Corollary 1.8 and Fact 1.11, is ultradisconnected and satisfies the hypotheses of Lemma 3.2. Let $\theta = \{x \in X$: there is no nowhere dense subset A of X such that $x \in \bar{A} \setminus A\}$. Then θ is clearly nodec.

We wish to show that θ is nonempty and has no isolated points, for then, by Fact 1.5, it will be ultradisconnected. To this end, we first show that $\theta = \{x \in X$: there is no discrete subset D of X such that $x \in \bar{D} \setminus D\}$. First notice that since X has no isolated points, every discrete subset of X is nowhere dense. Next let A be any nowhere dense subset of X and let $D = \{x \in A$: x is isolated in $A\}$. Clearly, D is discrete. We claim that D is dense in A, that is, that $A \subseteq \bar{D}$. For assume not. Then $E = A \setminus \bar{D}$ is a nonempty subset of X which has no isolated points. But $X \setminus E$ has no isolated points since $X \setminus E$ is dense in X. However, since X is ultradisconnected, this implies that E is open which contradicts the fact that E is a subset of the nowhere dense set A.

It now follows from Lemma 3.2 that the complement of θ is nowhere dense in X. Hence, θ is dense in X, so it has no isolated points. □

4. On $\leqslant$two-to-one images of ω^*

Since ω^* is big and since two-to-one maps cannot make a big space small, it is natural to ask whether ω^* admits a continuous two-to-one map onto a separable space. In this section, we obtain a partial answer to this question Ronnie Levy has asked us. For $n \in \mathbb{N}$, we call a function $f: X \to Y$

- *exactly n-to-one* if $|f^{\leftarrow}\{y\}| = n$ for all $y \in Y$.
- *$\leqslant$n-to-one* if $|f^{\leftarrow}\{y\}| \leqslant n$ for all $y \in Y$.

In what follows, all images are under maps, i.e., under continuous functions; expressions like $\leqslant$two-to-one image are self-explanatory.

We do not know if ω^* has a separable exactly two-to-one image. (This is genuinely harder, since $\mathbb{I}$, the closed interval obviously has a $\leqslant$two-to-one image, e.g. itself, but does not have an exactly two-to-one image, [3]; see also [5].) However, $\leqslant$two-to-one images are possible.

Theorem 4.1. *ω^* has a separable $\leqslant$ two-to-one image.*

When proving this theorem we will also find a sufficient condition for the existence of $\leqslant$two-to-one images with density κ of $U(\kappa)$, the space

$$U(\kappa) = \{p \in \beta\kappa : (\forall L \in [\kappa]^{<\kappa})[p \notin \bar{L}]\}$$

of uniform ultrafilters on κ:

Theorem 4.2. *A sufficient condition for the existence of $\leqslant$two-to-one images with density κ of $U(\kappa)$ is that there be a κ-crowded perfectly disconnected space of cardinality κ.*

We prove this later. It has a minor advantage of being even more general, and of allowing us to study images of $U_\lambda(\kappa)$, the space

$$U_\lambda(\kappa) = \{p \in \beta\kappa : (\forall L \in [\kappa]^{<\lambda})[p \notin \bar{L}]\}$$

of λ-uniform ultrafilters on κ. Once Theorem 4.2 is proved, Theorem 4.1 is immediate because of Example 3.3.

Because of our ignorance about $U_\lambda(\kappa)$ our only hope to construct a $\leqslant$two-to-one map from $U_\lambda(\kappa)$ onto a space B of density κ is to construct a $\leqslant$two-to-one map ϕ from $\beta\kappa$ onto B such that $\phi | U_\lambda(\kappa)$ maps $U_\lambda(\kappa)$ onto B. Since $Y = \phi^{\to}\kappa$ is a dense subset of B we think of B as a compactification bY of Y. We will find a necessary and sufficient condition on a space Y such that

(1) Y has a compactification bY such that there is a $\leqslant$two-to-one map ϕ from $\beta\kappa$ onto bY such that $\phi | \kappa$ is a bijection $\kappa \to Y$ and such that $\phi^{\to} U_\lambda(\kappa) = bY$.

We first argue that we may assume without loss of generality that $bY = \beta Y$ in (1). Indeed, there are maps $\psi : \beta\omega \to \beta Y$ and $\iota : \beta Y \to bY$ such that ψ extends $\phi | Y$ and such that ι extends id_Y. Then $\iota \circ \psi = \phi$ since $(\iota \circ \psi) | Y = \phi | Y$. Hence ψ is $\leqslant$two-to-one since ϕ is $\leqslant$two-to-one. Since all bijections $\kappa \to Y$ are equivalent, it follows that (1) is equivalent to

(2) If ϕ is a map $\beta\kappa \to \beta Y$ such that $\phi|\kappa$ is a bijection $\kappa \to Y$, then ϕ is $\leq$two-to-one and $\phi^{\leftarrow} U_\lambda(\kappa) = \beta Y$.

We next argue that (2) is equivalent to

(3) If ϕ is any map $\beta\kappa \to \beta Y$ such that $\phi|\kappa$ is a bijection $\kappa \to Y$, then

($\geq$) $(\forall y \in Y)(\exists^{\geq 1} p \in U_\lambda(\kappa))[\phi(p) = y]$, and

($\leq$) $(\forall y \in Y)(\exists^{\leq 1} p \in \kappa^*)[\phi(p) = y]$.

(Note that in the special case $\lambda = \omega$ ($\geq$) and ($\leq$) collapse to

$$(\forall y \in Y)(\exists! p \in \kappa^*)[\phi(p) = y].) \tag{$*$}$$

That (2) $\Rightarrow$ (3) is clear. To prove (3) $\Rightarrow$ (2) first note that ($\geq$) implies $\phi^{\leftarrow} U_\lambda(\kappa) = bY$. Next note that if $X = \phi^{\leftarrow} Y$ and $f = \phi|X$ then f is a closed $\leq$two-to-one map from X onto Y. As $\beta X = \beta\kappa$, since $\kappa \subseteq X \subseteq \beta\kappa$, so that $\phi = \beta f$, it now follows from the following observation that ϕ is $\leq$two-to-one.

Lemma 4.3. *Let $n \in \mathbb{N}$, let f be a closed map from X onto Y, and assume Y is normal. Then f is $\leq n$-to-one iff βf is $\leq n$-to-one.*

Proof. *Sufficiency* is clear. To prove *necessity* consider any set F of $n+1$ points of βX. There is a pairwise disjoint indexed collection $\langle A_x : x \in X\rangle$ of closed sets in X such that

(a) $(\forall x \in F)[x \in \mathrm{cl}_{\beta X} A_x]$, hence such that $(\forall x \in F)[\beta f(x) \in \mathrm{cl}_{\beta Y} f^{\rightarrow} A_x]$.

Indeed, let $\langle U_x : x \in F\rangle$ be an indexed collection of open sets in βX such that

$$(\forall x \in F)[x \in \mathrm{cl}_{\beta X} U_x] \quad \text{and} \quad (\forall x \neq y \in F)[\mathrm{cl}_{\beta X} U_x \cap \mathrm{cl}_{\beta X} U_y = \emptyset],$$

and for $x \in F$ let $A_x = X \cap \mathrm{cl}_{\beta X} U_x$.

Since $|F| = n+1$ and since f is $\leq n$-to-one, $\bigcap_{x \in F} f^{\rightarrow} A_x = \emptyset$. Since the $f^{\rightarrow} A_x$ are closed, because f is closed, and since βY is normal it follows that

(b) $\bigcap_{x \in F} \mathrm{cl}_{\beta Y} f^{\rightarrow} A_x = \emptyset$.

It follows from (a) and (b) that there are $x \neq y \in F$ such that $\beta f(x) \neq \beta f(y)$. $\square$

We now find separate characterizations for the ($\geq$)-part and for the ($\leq$)-part of (3). The ($\geq$)-part is simple. The ($\leq$)-part leads to another characterization of perfectly disconnected spaces.

Theorem 4.4. *If ϕ is any map $\beta\kappa \to \beta Y$ such that $\phi|\kappa$ is a bijection $\kappa \to Y$, then for every $y \in Y$*

$$\min\{\lambda \in [\omega, \kappa]: y \in \phi^{\rightarrow} U_\lambda(\kappa)\} = \min\{|L|: L \subseteq Y \setminus \{y\} \textit{ and } y \in \bar{L}\}.$$

(We do not define min $\emptyset$; if one minimum is not defined, then neither is the other.)

Theorem 4.5. *If ϕ is any map $\beta\kappa \to \beta Y$ such that $\phi|\kappa$ is a bijection $\kappa \to Y$, then*

$$(\forall y \in Y)(\exists^{\leq 1} p \in \kappa^*)[\phi(p) = y] \quad \textit{iff} \quad Y \textit{ is perfectly disconnected.}$$

Proof of Theorems 4.4 and 4.5. We first observe that the fact that ϕ is a continuous closed function $\beta\kappa \to \beta Y$ implies that:

$$(\forall y \in Y)(\forall K \subseteq \kappa)[\bar{K} \cap \phi^{\leftarrow}\{y\} \neq \emptyset \Leftrightarrow \overline{y \in \phi^{\rightarrow} K}]. \qquad (*)$$

Since $\phi|\kappa$ is a bijection $\kappa \to Y$ and since $\bar{K} \cap \kappa^* = \overline{K \backslash \{\xi\}} \cap \kappa^*$ for all $\xi \in \kappa$, it follows that

$$(\forall y \in Y)(\forall K \subseteq \kappa)[\bar{K} \cap \kappa^* \cap \phi^{\leftarrow}\{y\} \neq \emptyset \Leftrightarrow y \in \overline{(\phi^{\rightarrow} K) \backslash \{y\}}]. \qquad (*)$$

Proof of Theorem 4.4: This follows easily from (**) once one observes that

$$\min\{\lambda \in [\omega, \kappa]: U_\lambda(\kappa) \cap \phi^{\leftarrow}\{y\} \neq \emptyset\} = \min\{|L|: L \subseteq \kappa \text{ and } \bar{L} \cap \phi^{\leftarrow}\{y\} \neq \emptyset\}.$$

Proof of Theorem 4.5: Fix $y \in Y$ and use (**) to see that for arbitrary $y \in Y$

$$\begin{aligned} &(\exists p \neq q \in \kappa^*)[\phi(p) = \phi(q) = y] \\ \Leftrightarrow \quad &(\exists P, Q \subseteq \kappa)[P \cap Q = \emptyset \text{ and } \bar{P} \cap \kappa^* \cap \phi^{\leftarrow}\{y\} = \emptyset \text{ and } \\ &\qquad \bar{Q} \cap \kappa^* \cap \phi^{\leftarrow}\{y\} = \emptyset] \\ \Leftrightarrow \quad &(\exists A, B \subseteq Y)[A \cap B = \emptyset \text{ and } y \in \overline{A \backslash \{y\}} \text{ and } y \in \overline{B \backslash \{y\}}]. \quad \square \end{aligned}$$

Combining all of this we get:

Theorem 4.6. *For a space Y of cardinality κ the following are equivalent:*

(a) *If ϕ is a map $\beta\kappa \to \beta Y$ such that $\phi|\kappa$ is a bijection $\kappa \to Y$, then ϕ is $\leqslant$two-to-one and $\phi^{\rightarrow} U(\kappa) = \beta Y$; and*

(b) *Y is a κ-crowded perfectly disconnected space.*

Proof of Theorem 4.2. If Y is a κ-crowded space of cardinality κ, then there is a $\leqslant$two-to-one map from $U(\kappa)$ onto βY by Theorem 4.6. Clearly $d(\beta Y) \leqslant |Y| = \kappa$. Furthermore, $d(Y) = \kappa$ since Y is a κ-crowded perfectly disconnected space of cardinality κ. However, this does not rule out the possibility that $d(\beta Y) < \kappa$, cf. [1]. We ensure $d(\beta Y) \geqslant \kappa$ by our choice of Y: If there is a κ-crowded perfectly disconnected space of cardinality κ, then there is such a space that has a cellular family of cardinality κ: Let Y be a topological sum. Then βY also has a cellular family of cardinality κ, so $d(\beta Y) \geqslant \kappa$. $\square$

For our further study of $\leqslant$two-to-one images of $U(\kappa)$ we make the following observation:

Lemma 4.7. *Let Y be a perfectly disconnected crowded space and let ϕ be a map $\phi: \beta\kappa \to \beta Y$ such that $\phi|\kappa$ is a bijection $\kappa \to Y$. If $Z = \kappa^* \cap \phi^{\leftarrow} Y$, then $\phi|\mathrm{cl}_{\beta\kappa} Z$ is a homeomorphism $\mathrm{cl}_{\beta\kappa} Z \to \beta Y$.*

Proof. Since Y is crowded, $\phi^{\leftarrow}Z = Y$ by Theorem 4.4. Also, since Y is perfectly disconnected, $\phi|Z$ is one-to-one by Theorem 4.5. Moreover, since ϕ is closed, so is $\phi|\phi^{\leftarrow}Y$. But Z is closed in $\phi^{\leftarrow}Y$ since κ^* is closed in $\beta\kappa$. This shows $\phi|Z$ is a homeomorphism $Z \to Y$. As βY is the biggest compactification of Y it follows that $\phi|\bar{Z}$ is a homeomorphism $\bar{Z} \to \beta Y$. □

We are now ready to characterize the "multiple" points of βY.

Theorem 4.8. *Let Y be a perfectly disconnected crowded space and let ϕ be a map $\phi : \beta\kappa \to \beta Y$ such that $\phi|\kappa$ is a bijection $\kappa \to Y$. The following are equivalent for $y \in Y^*$:*

(a) $|\phi^{\leftarrow}\{y\}| = 2$; *and*

(b) *there is a discrete C^*-embedded (closed) subspace B of the space Y such that $y \in \mathrm{cl}_{\beta Y} B$.*

Proof. As in Lemma 4.7, let $Z = \kappa^* \cap \phi^{\leftarrow} Y$, so $\phi|\bar{Z}$ is a homeomorphism $\bar{Z} \to \beta Y$. Hence we may assume without loss of generality $\beta Y \subseteq \kappa^*$. Then ϕ is a retraction, and (a) is equivalent to

(A) $(\exists p \in \kappa^*)[p \neq y$ and $\phi(p) = y]$.

(A) $\Rightarrow$ (b): Let $p \in \kappa^*$ be such that $p \neq y$ and $\phi(p) = y$. Since $\phi|\beta Y = \mathrm{id}_{\beta Y}$, $p \notin \beta Y$. Hence there is $A \subseteq \kappa$ with $p \in \bar{A}$ and $\bar{A} \cap \beta Y = \emptyset$. As $\phi^{\leftarrow}\beta Y = \beta Y$ and ϕ is $\leq$two-to-one, it follows that $\phi|\bar{A}$ is one-to-one, hence it is a homeomorphism. It follows that $B = \phi^{\rightarrow}A$ is relatively discrete in Y. (Hence B is in fact closed discrete in Y since Y is perfectly disconnected.) Moreover, B is C^*-embedded since $\bar{A} = \beta A$.

(b) $\Rightarrow$ (A): Let A be a subset of κ such that $B = \phi^{\rightarrow}A$ is a discrete C^*-embedded subset of the space Y with $y \in \bar{B}$. Since $\bar{B} = \beta B$, $\phi|\bar{A}$ is a homeomorphism $\bar{A} \to \bar{B}$. Let $p = \phi(y)$. It remains to show $p \neq y$. To this end we prove:

$$\text{If } K \subseteq \kappa, \text{ then no homeomorphism } \bar{K} \to \kappa^* \text{ has a fixed point.} \qquad (*)$$

This is a routine consequence of the special case $K = \kappa$ of (*), which is a theorem of Frolík, [2, Theorem B]. So consider any $K \subseteq \kappa$, any embedding $e : \bar{K} \to \kappa^*$ and any $p \in \bar{K}$. We must prove $e(p) \neq p$. Without loss of generality assume that $p \in \kappa^*$, that $p \notin S$ for $S \subseteq K$ with $|S| < |K|$, and that $e(p) \in \bar{K}$. Since $\bar{K}$ is open and $e(p) \in e^{\rightarrow}K$, if $B = \bar{K} \cap e^{\rightarrow}K$ then $e(p) \in \bar{B}$. Since e is an embedding, our assumption that $p \notin S$ for $S \subseteq K$ with $|S| < |K|$ implies that $|B| = |K|$. There is $A \subseteq e^{\leftarrow}B$ with $|A| = |B \setminus e^{\rightarrow}A| = |K|$ such that $p \in \bar{A}$. Then $|K \setminus A| = |K| = |B \setminus e^{\rightarrow}A|$, hence there is a homeomorphism $h : \bar{K} \to \bar{B}$ such that $h|\bar{A} = e|\bar{A}$. As $B \subseteq \bar{K}$, h is an embedding $\bar{K} \to K^*$. Hence h has no fixed points by Frolík's theorem. But clearly $e(p) = h(p)$ since $h|\bar{A} = e|\bar{A}$ and since $p \in \bar{A}$. □

Recall that for a space Y one calls a point p of Y^* a *far point* of Y if there is no closed discrete set D in Y such that $p \in \mathrm{cl}_{\beta Y} D$. We state the special case $\kappa = \omega$ of Theorem 4.8 separately because we need it below.

Theorem 4.9. *Let Y be a countable perfectly disconnected crowded space and let ϕ be a map $\phi:\beta\omega\to\beta Y$ such that $\phi|\omega$ is a bijection $\omega\to Y$. Then for each $y\in Y^*$, we have:*

$$|\phi^{\leftarrow}\{y\}|=2 \quad \textit{iff} \quad y \textit{ is a far point of } Y.$$

We now turn to the question of when ω_1^* has a separable $\leq$two-to-one image. (Again, exactly two-to-one images seem hopelessly out of reach.) This is another question of Ronnie Levy, who also pointed out the following:

Fact 4.10. *If ω_1^* or $U(\omega_1)$ admits a $\leq$two-to-one map onto a separable space B, then $2^{\omega_1}=2^{\omega}$.*

Proof. Let X denote ω_1^* or $U(\omega_1)$, and let f be a $\leq$two-to-one map from X onto a separable space B. Let C be a countable subset of X such that $f^{\to}C$ is dense in B. Then $f^{\to}\bar{C}=B$, hence $f|(X\setminus\bar{C})$ is one-to-one. But $U(\omega_1)$ embeds in $X\setminus\bar{C}$. Hence f induces an embedding of $U(\omega_1)$ into B. Therefore, $2^{\omega_1}=w(U(\omega_1))\leq w(B)\leq 2^{d(B)}=2^{\omega}$. □

In the other direction, we can only prove:

Fact 4.11. *If $2^{\omega_1}=2^{\omega}$, then*

(a) *there is a $\leq$three-to-one map from ω_1^* onto a separable space,*

(b) *if there is a countable crowded perfectly disconnected space that has a far point, then there is a $\leq$two-to-one map from ω_1^* onto a separable space.*

Proof. Let Y be a crowded perfectly disconnected space, and let ϕ be a map $\beta\omega\to\beta Y$ such that $\phi|\omega$ is a bijection $\omega\to Y$. We already know that ϕ is $\leq$two-to-one and that $\phi^{\to}\omega^*=\beta Y$.

Since ω_1^* is homeomorphic to $\omega^*+\omega_1^*$, we prove (a) if we find an embedding $\omega_1^*\to\beta Y$ and we prove (b) if we find an embedding of ω_1^* into the far points of Y.

As is well known, ω_1^* embeds into $\beta\omega$ if (and only if) $2^{\omega_1}=2^{\omega}$. Therefore we prove (a) by observing that $\beta\omega$ embeds into βY. And we prove (b) if we prove that $\beta\omega$ embeds into the far points of Y. If every crowded (perfectly disconnected) space has far points then this is true. If we know only that there is a countable crowded space that has a far point (but perhaps only one far point) we must choose Y carefully:

If S is a countable crowded perfectly disconnected space that has a far point, then $Y=S\times\omega$ is a countable crowded perfectly disconnected space that has a countable relatively discrete set A of far points with the property that $\mathrm{cl}_{\beta Y} A\subseteq Y^*$. By the proof of [6, 3.1], every point of $\mathrm{cl}_{\beta Y} A$ is far. Also, since $Y\cup A$ is normal, being countable, and since A is closed in $Y\cup A$, $\mathrm{cl}_{\beta Y} A=\beta A\approx\beta\omega$. □

Note that this trick does not help us to find a finite-to-one map from $U(\omega_1)$ onto a separable space.

5. On *n*-irresolvable spaces

Definition 5.1. If $n \in \omega$, a space X is *n-irresolvable* if $n = \max\{|\lambda|: \lambda$ is a pairwise disjoint collection of dense subsets of $X\}$.

Notice that with this definition, irresolvability is equivalent to 1-irresolvability. Thus, we know how to construct 1-irresolvable regular spaces. We now wish to find n-irresolvable spaces for each $n \geq 2$.

Theorem 5.2. *For every $n \in [2, \omega)$ there is a (necessarily crowded) n-irresolvable countable regular space.*

Proof. Clearly, if $k \leq n$ and there is an n-irresolvable countable regular space X, then there is a k-irresolvable countable regular space Y: Fix n pairwise disjoint dense subsets of X and let Y be the union of k of them. Hence it suffices to find a 2^n-irresolvable space for each $n \in [1, \omega)$.

Let $\mathcal{I}$ be an infinite maximal independent family of subsets of ω such that for each p, q in ω with $p \neq q$ the set $\{I \in \mathcal{I}: |\{p, q\} \cap I| = 1\}$ is infinite. Let τ be the topology on ω which has $\mathcal{I} \cup \{\omega \setminus I: I \in \mathcal{I}\}$ as a subbase. Then $X = \langle \omega, \tau \rangle$ is a regular irresolvable space: It is Hausdorff because each pair of points can be separated by a subbasic open set by our choice of $\mathcal{I}$, it is regular because each subbasic open set is also closed, and it is irresolvable because of the maximality of $\mathcal{I}$. Therefore, by Fact 3.1 and the observation that X is zero-dimensional there is a clopen subspace U of X such that U is hereditarily irresolvable. Let $\mathcal{A}$ and $\mathcal{B}$ be disjoint finite subsets of $\mathcal{I}$ such that $\bigcap \mathcal{A} \setminus \bigcup \mathcal{B} \subseteq U$. Then $\mathcal{I}' = (\mathcal{I} \setminus (\mathcal{A} \cup \mathcal{B})) | U$ is an independent family of subsets such that $\tau' = \mathcal{I}' \cup \{U \setminus I: I \in \mathcal{I}'\}$ generates a hereditarily irresolvable topology on U which is still regular because $\mathcal{A}$ and $\mathcal{B}$ are finite.

Now fix a positive natural number n, and let $\mathcal{F} \subseteq \mathcal{I}'$ be such that $|\mathcal{F}| = n$. Let $\mathcal{S}$ be the topology on U that has $(\mathcal{I}' \setminus \mathcal{F}) \cup \{U \setminus I: I \in \mathcal{I}' \setminus \mathcal{F}\}$ as a subbase. Clearly

$$\mathcal{D} = \{\bigcap \mathcal{G} \setminus \bigcup (\mathcal{F} \setminus \mathcal{G}): \mathcal{G} \subseteq \mathcal{F}\}$$

is a pairwise disjoint collection of dense subsets of $\langle U, \mathcal{S} \rangle$. Let $\mathcal{K}$ be a pairwise disjoint collection of $\geq 2^n + 1$ subsets of U. We show not all members of $\mathcal{K}$ are dense:

For each nonempty clopen $V \subseteq U$, not all members of $\mathcal{K}$ are dense in $\langle V, \tau' | V \rangle$ because $\langle U, \tau' \rangle$ is hereditarily irresolvable, so there is a nonempty clopen $W \subseteq U$ that misses at least one member of $\mathcal{K}$. Continuing this process we find a nonempty clopen set W that misses all but at most one member of $\mathcal{K}$.

So if $D \in \mathcal{D}$ we can find a nonempty clopen $D' \subseteq D$ that misses all but at most one member of $\mathcal{K}$; we may assume that D' is basic open, and as D has the form $\bigcap \mathcal{G} \setminus \bigcup (\mathcal{F} \setminus \mathcal{G})$ for some $\mathcal{G} \subseteq \mathcal{F}$, D' will have the form $D \cap ((\bigcap \mathcal{A}) \setminus \bigcup \mathcal{B})$ for disjoint finite $\mathcal{A}, \mathcal{B} \subseteq \mathcal{I}' \setminus \mathcal{F}$. For a $C \in \mathcal{D} \setminus \{D\}$ we can repeat this argument with the set $C \cap (\bigcap \mathcal{A} \setminus \bigcup \mathcal{B})$ to find disjoint finite $\mathcal{A}', \mathcal{B}' \subseteq \mathcal{I}' \setminus (\mathcal{F} \cup \mathcal{A} \cup \mathcal{B})$ such that $(C \cap \bigcap \mathcal{A} \setminus \bigcup \mathcal{B}) \cap (\bigcap \mathcal{A}') \setminus \bigcup \mathcal{B}' = C \cap (\bigcap (\mathcal{A} \cup \mathcal{A}') \setminus \bigcup (\mathcal{B} \cup \mathcal{B}'))$ misses all but at most one member of $\mathcal{F}$. Since $\mathcal{D}$ is finite, we end up finding disjoint finite $\mathcal{L}, \mathcal{M} \subseteq \mathcal{I}' \setminus \mathcal{F}$ such that

$$(\forall D \in \mathcal{D})[D \cap \bigcap \mathcal{L} \setminus \bigcup \mathcal{M} \text{ misses all but at most one member of } \mathcal{K}].$$

Since $\mathcal{D}$ covers U it follows that $\bigcap \mathcal{L} \setminus \bigcup \mathcal{M}$ misses all but at most $|\mathcal{D}|$ member of $\mathcal{K}$. As $|\mathcal{D}| < |\mathcal{K}|$ this shows there is $K \in \mathcal{K}$ with $(\bigcap \mathcal{L} \setminus \bigcup \mathcal{M}) \cap K = \emptyset$. Of course, $\bigcap \mathcal{L} \setminus \bigcup \mathcal{M}$ is a nonempty open subset of $\langle U, \mathcal{S} \rangle$. □

References

[1] W. Comfort, An example in density character, Arch. Math. 14 (1963) 422-423.

[2] Z. Frolík, Fixed points of maps of $\beta\mathbb{N}$, Bull. Amer. Math. Soc. 74 (1968) 187-191.

[3] O.G. Harrold, The non-existence of a certain type of transformation, Duke Math. J. 5 (1939) 789–793.

[4] E. Hewitt, A problem of set-theoretic topology, Duke Math. J. 10 (1943) 309-333.

[5] J. Mioduszewski, On two-to-one continuous functions, Dissertationes Math. (Rozprawy Mat.) 24 (1961) 24.

[6] E.K. van Douwen, Why certain Čech-Stone remainders are not homogeneous, Colloq. Math. 41 (1979) 45-52.