WEIRD \mathbb{R} -FACTORIZABLE GROUPS

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ABSTRACT. The problem of the existence of non-pseudo- \aleph_1 -compact \mathbb{R} -factorizable groups is studied. It is proved that any such group is submetrizable and has weight larger than ω_1 . Closely related results concerning the \mathbb{R} -factorizability of products of topological groups and spaces are also obtained (a product $X \times Y$ of topological spaces is said to be \mathbb{R} -factorizable if any continuous function $X \times Y \to \mathbb{R}$ factors through a product of maps from X and Y to second-countable spaces). In particular, it is proved that the square $G \times G$ of a topological groups G is \mathbb{R} -factorizable as a group if and only if it is \mathbb{R} -factorizable as a product of spaces, in which case G is pseudo- \aleph_1 -compact. It is also proved that if the product of a space X and an uncountable discrete space is \mathbb{R} -factorizable, then X^ω is heredirarily separable and heredirarily Lindelöf.

In the middle of the past century Pontryagin proved that any continuous function on a compact topological group factors through a continuous homomorphism to a second-countable group (see, e.g., [7, Example 37]). This result gave rise to the theory of \mathbb{R} -factorizable groups, which has been fruitfully developed since then.

Definition ([15]). A topological group G is said to be \mathbb{R} -factorizable if any continuous function $f: G \to \mathbb{R}$ factors through a homomorphism to a second-countable group, i.e., there exists a second-countable topological group H, a continuous homomorphism $h: G \to H$, and a continuous function $g: H \to \mathbb{R}$ for which $f = g \circ h$.

The notion of an \mathbb{R} -factorizable group was explicitly introduced by Tkachenko [15], who also obtained the first fundamental results. Among other things he proved that the following topological groups are \mathbb{R} -factorizable:

• any Lindelöf group;

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- any totally bounded group, that is, a group G such that, given any open neighborhood U of its identity element, there exists a finite subset A of G for which AU = G (totally bounded groups are precisely subgroups of compact groups);
- any subgroup of a Lindelöf Σ -group, in particular, any subgroup of a σ -compact group;
- any dense subgroup of an arbitrary product of Lindelöf Σ -groups.

In the decades since Tkachenko's paper [15] was published, the theory of \mathbb{R} -factorizable groups has been extensively developed; it is surveyed in Chapter 8 of the book [1].

However, many problems concerning \mathbb{R} -factorizable groups remain open. We consider the four questions posed below (see also [15]) to be the most important of them. Recall that a topological space X is said to be $pseudo-\aleph_1$ -compact if any locally finite (or, equivalently, any discrete) family of open sets in X is at most countable.

Question 1. Is any \mathbb{R} -factorizable group pseudo- \aleph_1 -compact?

Question 2. Is the image of an \mathbb{R} -factorizable group under a continuous homomorphism \mathbb{R} -factorizable?

Note that Question 2 is equivalent to the question of whether the image of an \mathbb{R} -factorizable group under a continuous isomorphism \mathbb{R} -factorizable, because the quotients of \mathbb{R} -factorizable groups are \mathbb{R} -factorizable [1, Theorem 8.4.2].

Question 3. Is the square of an \mathbb{R} -factorizable group \mathbb{R} -factorizable?

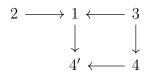
Question 4. Is the property of being \mathbb{R} -factorizable topological in the class of topological groups? In other words, is any topological group homeomorphic to an \mathbb{R} -factorizable one \mathbb{R} -factorizable?

If H is a topological group and D is a discrete uncountable topological group, then the group $H \times D$ is not \mathbb{R} -factorizable, because it is not ω -narrow. Thus, the following question is of interest in relation to Question 4.

Question 4'. Is it true that no \mathbb{R} -factorizable group is homeomorphic to a product $H \times D$, where H is a topological group and D is an uncountable discrete space?

The first question is most intriguing, at least because if the answer to it is negative, then so are the answers to Questions 2 and 3. We prove in this paper that the above questions are related as shown in

the following diagram. An arrow $A \to B$ means that if the answer to Question A is positive, then so is the answer to Question B.



According to Theorem 8.5.2 of [1], a topological group G is \mathbb{R} -factorizable and pseudo- \aleph_1 -compact if and only if it is m-factorizable, that is, any continuous map $f: G \to M$ to any metrizable space M factors through a continuous homomorphism to a second-countable topological group. A class of m-factorizable groups is very important; see Section 8.5 of [1]. Question 1 can be formulated as follows: Is any \mathbb{R} -factorizable group m-factorizable?

As the question of the existence of non-pseudo- \aleph_1 -compact \mathbb{R} -factorizable groups is so important, we give these groups a name.

Definition. An \mathbb{R} -factorizable group which is not pseudo- \aleph_1 -compact is called a *weird* \mathbb{R} -factorizable group.

Yet another way to state Question 1 is: Is it true that weird \mathbb{R} -factorizable groups do not exist?

Weird \mathbb{R} -factorizable groups have rather abnormal properties. In this paper we obtain results which imply the following theorem.

Theorem A. Let G be a weird \mathbb{R} -factorizable group. Then

- (1) $G \times G$ is not \mathbb{R} -factorizable;
- (2) there exists a surjective continuous homomorphism of G to a non- \mathbb{R} -factorizable group;
- (3) $w(G) > \omega_1$;
- (4) $w(G)^{\omega} > 2^{\omega_1}$;
- (5) $\psi(G) \leq \omega$, that is, G is submetrizable;
- (6) if H is a topological group and the group $G \times H$ is \mathbb{R} -factorizable, then
 - (a) H^{ω} is hereditarily Lindelöf and hereditarily separable;
 - (b) if $w(H) < \omega_1$, then H is second-countable;
 - (c) under CH, H is second-countable.

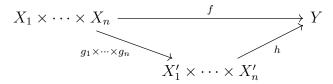
Assertion (1) follows from Exercise 8.5.a in [1] (see also Corollary 4.1 in the present paper), and assertion (4) follows from Theorem 8.5.8 in [1]. The other assertions of the theorem are new; their proofs are given in Section 4.

It is seen from Theorem A that the problem of the existence of weird \mathbb{R} -factorizable groups is closely related to the \mathbb{R} -factorizability

of products of groups. Only recently has the fundamental question of the multiplicativity of the class of \mathbb{R} -factorizable groups been answered by constructing Lindelöf (and hence \mathbb{R} -factorizable) groups G and H whose product $G \times H$ is not \mathbb{R} -factorizable [8, 13]¹; moreover, one of these groups can be made second-countable. Section 3 of the present paper is devoted to \mathbb{R} -factorizable products of groups.

The key role in the study of \mathbb{R} -factorizable products of groups is played by the notion of an \mathbb{R} -factorizable product of spaces.

Definition ([10, 8]). Given topological spaces X_1, \ldots, X_n and Y, we say that a map $f: X_1 \times \cdots \times X_n \to Y$ is \mathbb{R} -factorizable if it factors through a product of continuous maps to second-countable spaces, that is, if there exist second-countable spaces X'_1, \ldots, X'_n and continuous maps $g_i: X_i \to X'_i$, $i \leq n$, and $h: X'_1 \times \cdots \times X'_n \to Y$ such that $f = h \circ (g_1 \times \cdots \times g_n)$, i.e., the following diagram is commutative:



We say that a product $X_1 \times \cdots \times X_n$ is \mathbb{R} -factorizable (or multiplicatively \mathbb{R} -factorizable, when there is a danger of confusion) if any continuous function $f: X_1 \times \cdots \times X_1 \to \mathbb{R}$ is \mathbb{R} -factorizable.

The notion of an \mathbb{R} -factorizable map was introduced in [10], where it was essentially proved that, for the free topological group F(S) of the Sorgenfrey line S, the product $F(S) \times F(S)$ is not multiplicatively \mathbb{R} -factorizable. In turn, the notion of a multiplicatively \mathbb{R} -factorizable product was introduced in [8]; in the same paper, the following statement was proved, which is the main tool for constructing topological groups whose products are not \mathbb{R} -factorizable (as groups).

Theorem B ([8, Corollary 2(2)]). If the product $G \times H$ of topological groups G and H is an \mathbb{R} -factorizable group, then $G \times H$ is a multiplicatively \mathbb{R} -factorizable product.

In this paper, we refine Theorem B as follows (see Theorem 3.1 in Section 3).

Theorem C. For topological groups G and H, the following conditions are equivalent:

(1) the group $G \times H$ is \mathbb{R} -factorizable;

¹As shown in [8], the products of Lindelöf groups constructed in [14, 12] are not \mathbb{R} -factorizable either.

(2) G and H are \mathbb{R} -factorizable and the product $G \times H$ is multiplicatively \mathbb{R} -factorizable.

In [9] the notion of \mathbb{R} -factorizability was extended from the class of topological groups to the much larger class of topological universal algebras. Results of [9] have enabled us to prove the following statement (see Theorem 3.2 below).

Theorem D. Let G be a topological group. Then the group $G \times G$ is \mathbb{R} -factorizable if and only if the product $G \times G$ is multiplicatively \mathbb{R} -factorizable.

Corollary. Given a topological group G, the \mathbb{R} -factorizability of the group $G \times G$ is a topological property of G. In other words, if the group $G \times G$ is \mathbb{R} -factorizable and H is a topological group homeomorphic to G, then the group $H \times H$ is \mathbb{R} -factorizable.

This corollary shows why a positive answer to Question 3 gives a positive answer to Question 4.

In [3] Blair and Hager considered conditions under which a product $X \times Y$ is z-embedded in $\beta X \times \beta Y$. We will show in Section 2, which is devoted to multiplicatively \mathbb{R} -factorizable products, that a product $X \times Y$ has this property if and only if it is \mathbb{R} -factorizable (see Proposition 2.1). In the same section we also prove the following statement (this is Theorem 2.1).

Theorem E. Suppose that a product $X \times Y$ is \mathbb{R} -factorizable and Y is not pseudo- \aleph_1 -compact. Then

- (1) X^{ω} is hereditarily Lindelöf and hereditarily separable;
- (2) if $w(X) \leq \omega_1$, then X is second-countable;
- (3) under CH, X is second-countable.

This theorem strengthens Proposition 2.1(b) of [3].

The CH assumption cannot be omitted from (3), because it has recently been shown by the authors jointly with Anton Lipin that if $\mathfrak{b} > \omega_1$, then the product of the countable Fréchet–Urysohn fan and a discrete space of cardinality ω_1 is \mathbb{R} -factorizable.

1. Preliminaries

All topological spaces and groups considered in this paper are assumed to be Tychonoff. Throughout the paper, by a space we mean a topological space and use I to denote an arbitrary index set. Given ordinals α and β , by $[\alpha, \beta]$ we denote the set of all ordinals γ satisfying the inequalities $\alpha \leq \gamma \leq \beta$. The weight of a space X is denoted

by w(X) and its pseudocharacter (that is, the least cardinal κ such that every point of X is the intersection of at most κ -many neighborhoods), by $\psi(X)$. By a $Lindel\"{o}f$ Σ -group we mean a topological group whose underlying topological space is a Lindel\"{o}f Σ -space, that is, can be represented as a continuous image of a perfect preimage of a second-countable space. A topological space X is said to be $perfectly \varkappa$ -normal if the closure of any open set in X is a zero set.

Definition 1.1. Given a cardinal κ , a topological space X is said to be $pseudo-\kappa-compact$ if the cardinality of any locally finite family of open sets in X is less than κ .

Remark 1.1. It is easy to see that a Tychonoff space X is pseudo- κ -compact if and only if the cardinality of any discrete family of cozero sets in X is less than κ .

Indeed, suppose that there exists a locally finite family $\{U_{\alpha} : \alpha \in \kappa\}$ of nonempty open sets in X. Let us show that there exists a discrete family of cozero sets which has cardinality κ . Choose points $x_{\alpha} \in X$ and cozero sets $V_{\alpha} \subset X$ so that $x_{\alpha} \in V_{\alpha} \subset U_{\alpha}$ and V_{α} intersects only finitely many sets U_{β} for each $\alpha < \kappa$. Let $f : \kappa \to \kappa$ be a function such that $V_{\alpha} \cap V_{\beta} = \emptyset$ for all $\beta \geq f(\alpha)$. Clearly, f is increasing and the family $\{V_{f(\alpha)} : \alpha \in \kappa\}$ is a locally finite disjoint (and hence discrete) family of cozero sets.

Definition 1.2. A topological group G is said to be ω -narrow if, for every open neighborhood V of the identity element in G, there exists a countable set $A \subset G$ such that VA = G (or, equivalently, such that AV = G).

According to [1, Proposition 8.1.3], every \mathbb{R} -factorizable group is ω -narrow.

Definition 1.3. A subspace Y of a topological space X is said to be z-embedded in X if, for every zero set Z in Y, there exists a zero set F in X such that $F \cap Y = Z$.

Clearly, "zero" in this definition can be replaced by "cozero." It is also clear that any z-embedded subspace Z of a z-embedded subspace Y of a space X is z-embedded in X.

The property of being z-embedded plays a crucial role in the theory of \mathbb{R} -factorizable groups, because an ω -narrow topological group G is \mathbb{R} -factorizable if and only if G is z-embedded in every topological group that contains G as a topological subgroup [1, Theorem 8.2.7].

Recall that a subspace Y of a space X is C-embedded (C*-embedded) in X if any continuous (any bounded continuous) function $f: Y \to \mathbb{R}$

has a continuous extension to X. Obviously, all C- and C^* -embedded subspaces of X are z-embedded. It is also known that a zero set Y of X is z-embedded in X if and only if Y is C^* -embedded in X [5, Corollary 11.7]. On the other hand, any cozero set of X is z-embedded in X [5, Lemma 11.12].

Remark 1.2. The union of any discrete family of cozero sets in a space X is z-embedded in X.

Indeed, let $\mathscr{U} = \{U_{\iota} : \iota \in I\}$ be a discrete family of cozero sets, and let $f_{\iota} : X \to [0, 1]$ witness their being cozero. The function

$$f = \sum_{\iota \in I} f_{\iota} \colon X \to \mathbb{R}$$

is well defined (because \mathscr{U} is disjoint) and continuous (because \mathscr{U} is discrete and hence locally finite). Clearly, $f^{-1}(\{0\}) = X \setminus \bigcup \mathscr{U}$. Thus, $\bigcup \mathscr{U}$ is a cozero set in X; therefore, it is z-embedded.

Remark 1.3. Let $\mathscr{U} = \{U_{\iota} : \iota \in I\}$ be a discrete family of nonempty cozero sets in a space X, and let $y_{\iota} \in U_{\iota}$ for each $\iota \in I$. Then the (discrete) set $Y = \{y_{\iota} : \iota \in I\}$ is C-embedded in X.

Indeed, let $f_{\iota} \colon X \to \mathbb{R}$ be continuous functions witnessing that the U_{ι} are cozero, and let $g \colon Y \to \mathbb{R}$ be any function. The function

$$f = \sum_{\iota \in I} \frac{g(y_{\iota})}{f_{\iota}(y_{\iota})} \cdot f_{\iota} \colon X \to \mathbb{R}$$

is well defined and continuous, and its restriction to Y coincides with g.

Recall that a topological space is *submetrizable* if it admits a coarser metrizable topology. In this paper we repeatedly use the following well-known theorems.

Theorem 1.1 (see, e.g., [1, Theorem 3.3.16]). A topological group is submetrizable if and only if it has countable pseudocharacter.

Theorem 1.2 ([1, Theorem 8.4.2]). The image of an \mathbb{R} -factorizable group under a quotient (= open) homomorphism is \mathbb{R} -factorizable.

2. R-Factorizable Products

Definition 2.1. A cozero rectangle in a product $X \times Y$ of topological spaces X and Y is any set of the form $V \times W$, where V and W are cozero sets in X and Y, respectively.

Proposition 2.1. For any spaces X and Y, the following conditions are equivalent:

(1) $X \times Y$ is \mathbb{R} -factorizable;

- (2) any cozero set in $X \times Y$ is a countable union of cozero rectangles;
- (3) $X \times Y$ is z-embedded in $\beta X \times \beta Y$;
- (4) if X is z-embedded in X' and Y is z-embedded in Y', then $X \times Y$ is z-embedded in $X' \times Y'$;
- (5) there exist spaces X' and Y' such that $X' \times Y'$ is \mathbb{R} -factorizable and $X \times Y$ is z-embedded in $X' \times Y'$;
- (6) there exist spaces X' and Y' such that $X' \times Y'$ is Lindelöf and $X \times Y$ is z-embedded in $X' \times Y'$.

Proof. First, we prove the implication $(1) \Rightarrow (2)$. Let U be a cozero set in $X \times Y$, and let f be a continuous function on $X \times Y$ such that $(X \times Y) \setminus U = f^{-1}(\{0\})$. Since $X \times Y$ is \mathbb{R} -factorizable, it follows that there exist second-countable spaces M and H, continuous maps $p \colon X \to M$ and $q \colon Y \to H$, and a continuous function $h \colon M \times H \to \mathbb{R}$ such that $f = h \circ (p \times q)$. The preimage $h^{-1}(\mathbb{R} \setminus \{0\})$ is a cozero set in the second-countable Tychonoff space $M \times H$; hence there exist cozero sets $V_i \subset M$ and $V_i \subset H$, $V_i \in \mathcal{U}$, such that $V_i \subset \mathcal{U}$ and $V_i \subset \mathcal$

Now we show that $(2) \Rightarrow (1)$. Let $\mathscr{B} = \{U_n : n \in \omega\}$ be a countable base of the topology of \mathbb{R} , and let $f: X \times Y \to \mathbb{R}$ be a continuous function. By assumption, for each $n \in \omega$,

$$f^{-1}(U_n) = \bigcup_{i \in \omega} (V_{n,i} \times W_{n,i}),$$

where the $V_{n,i}$ and $W_{n,i}$ are cozero sets in X and Y, respectively. Let $g_{n,i} \colon X \to \mathbb{R}$ and $h_{n,i} \colon Y \to \mathbb{R}$ be continuous functions for which $V_{n,i} = g_{n,i}^{-1}(\mathbb{R} \setminus \{0\})$ and $W_{n,i} = h_{n,i}^{-1}(\mathbb{R} \setminus \{0\})$. We set

$$g = \bigwedge_{n,i \in \omega} g_{n,i} \colon X \to \mathbb{R}^{\omega \times \omega}$$
 and $h = \bigwedge_{n,i \in \omega} h_{n,i} \colon Y \to \mathbb{R}^{\omega \times \omega}$.

We have

$$(g \times h)^{-1} ((g \times h)(f^{-1}(U_n))) = f^{-1}(U_n),$$

because $(x,y) \notin f^{-1}(U_n)$ if and only if

$$(g \times h)((x,y)) = ((a_{n,i})_{(n,i) \in \omega \times \omega}, (b_{n,i})_{(n,i) \in \omega \times \omega}),$$

where $a_{n,i} \cdot b_{n,i} = 0$ for all $i \in \omega$. Note that if $f((x,y)) \neq f((x',y'))$, then there exists an $n \in \omega$ for which $f((x,y)) \in U_n$ and $f((x',y')) \notin U_n$, whence $(g \times h)((x,y)) \neq (g \times h)((x',y'))$. Therefore, setting

$$\varphi((a,b)) = f(x,y)$$
, where (x,y) is any point in $(g \times h)^{-1}(\{(a,b)\})$,

we obtain a well-defined function $\varphi \colon (g \times h)(X \times Y) \to \mathbb{R}$. For each $n \in \omega$,

$$\varphi^{-1}(U_n) = \bigcup_{i \in \omega} \{ ((a_{n,i})_{(n,i) \in \omega \times \omega}, (b_{n,i})_{(n,i) \in \omega \times \omega}) \in (g \times h)(X \times Y) : a_{n,i} \cdot b_{n,i} \neq 0 \}.$$

All these sets are open in $(g \times h)(X \times Y)$ and hence φ is continuous. Clearly, $f = \varphi \circ (g \times h)$.

The equivalences $(2) \Leftrightarrow (3) \Leftrightarrow (4)$ follow from Theorem 1.1 of [3].

Let us prove (5) \Leftrightarrow (6). Since any cozero set is F_{σ} , Lindelöfness is preserved by F_{σ} subspaces, and any open set in a product of Tychonoff spaces is a union of cozero rectangles, it follows from (1) \Leftrightarrow (2) that all Lindelöf products are \mathbb{R} -factorizable. Therefore, (6) \Rightarrow (5). To see that (5) \Rightarrow (6), it suffices to note that if $X' \times Y'$ is \mathbb{R} -factorizable, then $X' \times Y'$ is z-embedded in $\beta X' \times \beta Y'$ (because (1) \Rightarrow (3)) and hence $X \times Y$ is z-embedded in $\beta X' \times \beta Y'$.

The implication $(3) \Rightarrow (6)$ is obvious. It remains to show that $(5) \Rightarrow (2)$. Let U be a cozero set in $X \times Y$, and let U' be a cozero set in an \mathbb{R} -factorizable product $X' \times Y'$ for which $U = U' \cap (X \times Y)$. Since (2) holds for $X' \times Y'$, it follows that $U' = \bigcup_{n \in \omega} (V_n \times W_n)$, where the V_n and W_n are cozero sets in X' and Y', respectively. Clearly, the $V_n \cap X$ and $W_n \cap Y$ are cozero sets in X and Y. We have

$$U = \left(\bigcup_{n \in \omega} (V_n \times W_n)\right) \cap (X \times Y) = \bigcup_{n \in \omega} \left((V_n \times W_n) \cap (X \times Y)\right)$$
$$= \bigcup_{n \in \omega} \left((V_n \cap X) \times (W_n \times Y)\right).$$

Thus, (2) holds for $X \times Y$.

The paper [3] studied pairs of spaces X and Y satisfying condition (3) in Proposition 2.1. According to this proposition, (3) is equivalent to the \mathbb{R} -factorizability of the product $X \times Y$. In what follows, when referring to [3], we will bear in mind the equivalence (1) \Leftrightarrow (3).

Given a cardinal κ , by $D(\kappa)$ we denote κ with the discrete topology; thus, $D(\kappa)$ is a discrete space of cardinality κ .

Proposition 2.2. If a product $X \times Y$ of spaces is \mathbb{R} -factorizable and Y contains a discrete family of open sets of cardinality κ , then the product $X \times D(\kappa)$ is \mathbb{R} -factorizable.

Proof. Let $\{U_{\alpha} : \alpha < \kappa\}$ be a discrete family of nonempty open sets in Y. We choose $y_{\alpha} \in U_{\alpha}$ for each $\alpha < \kappa$ and set $Q = \{y_{\alpha} : \alpha < \kappa\}$. Clearly, Q is homeomorphic to $D(\kappa)$.

We claim that $X \times Q$ is C-embedded in $X \times Y$. Indeed, let f be a continuous function on $X \times Q$. For each $\alpha < \kappa$, we choose a continuous function $g_{\alpha} \colon Y \to [0,1]$ such that $g_{\alpha}(Y \setminus U_{\alpha}) = \{0\}$ and $g_{\alpha}(y_{\alpha}) = 1$ and define a function $h_{\alpha} \colon X \times Y \to \mathbb{R}$ by setting $h_{\alpha}(x,y) = f(x,y_{\alpha}) \cdot g_{\alpha}(y)$ for $(x,y) \in X \times Y$. The function $h = \sum_{\alpha < \kappa} h_{\alpha}$ is a continuous extension of f.

Since $X \times Q$ is C-embedded in $X \times Y$, it follows that $X \times Q$ is z-embedded in $X \times Y$. According to Proposition 2.1, the product $X \times Q$ is \mathbb{R} -factorizable.

Proposition 2.3. For a space Y and a cardinal κ , the following conditions are equivalent:

- (1) $X \times D(\kappa)$ is \mathbb{R} -factorizable;
- (2) for any family $\{F_{\alpha} : \alpha < \kappa\}$ of zero sets in X, there exists a second-countable space M and a continuous map $g : X \to M$ such that $F_{\alpha} = g^{-1}(g(F_{\alpha}))$ and $g(F_{\alpha})$ is closed in M for each $\alpha < \kappa$;
- (3) every continuous map $f: X \to Y$ to a space Y with $w(Y) \le \kappa$ factors through a continuous map to a second-countable space.

Proof. Let us prove that $(1) \Rightarrow (2)$. For each $\alpha < \kappa$, we fix a continuous function f_{α} on X such that $f_{\alpha}^{-1}(\{0\}) = F_{\alpha}$. The function

$$f: X \times D(\kappa) \to \mathbb{R}, \quad (x, \alpha) \mapsto f_{\alpha}(x),$$

is continuous. Condition (1) implies the existence of second-countable spaces M and E and continuous maps $g\colon X\to M,\ q\colon D(\kappa)\to E,$ and $h\colon M\times E\to \mathbb{R}$ for which $f=h\circ (g\times q).$ For each $\alpha<\kappa,$ $F_{\alpha}\subset g^{-1}(g(F_{\alpha})).$ On the other hand, if $x\in g^{-1}(g(F_{\alpha})),$ then there exists a $y\in F_{\alpha}$ for which g(y)=g(x) and $f((x,\alpha))=h(g(x),q(\alpha))=h(g(y),q(\alpha))=f((y,\alpha))=0,$ which means that $f_{\alpha}(x)=0$ and $x\in F_{\alpha}.$ Thus, $F_{\alpha}=g^{-1}(g(F_{\alpha})).$ Note that, for $x\in M$ and $\alpha\in\omega_1,$ $h(x,q(\alpha))=0$ if and only if there exists a $z\in X$ for which g(z)=x and $f(z,\alpha)=0,$ i.e., $z\in F_{\alpha}$ and $x\in g(F_{\alpha}).$ Therefore, $g(F_{\alpha})\times\{q(\alpha)\}=h^{-1}(\{0\})\cap (M\times\{q(\alpha)\}).$ This set is closed in $M\times\{q(\alpha)\},$ and hence $g(F_{\alpha})$ is closed in M.

To prove $(2) \Rightarrow (3)$, we take a base $\{U_{\alpha} : \alpha < \kappa\}$ of Y consisting of cozero sets and put $F_{\alpha} = f^{-1}(Y \setminus U_{\alpha})$ for $\alpha < \kappa$. Let M and $g: X \to M$ be as in (2). Note that if $x, y \in Y$ and $f(x) \neq f(y)$, then $g(x) \neq g(y)$. Indeed, there exists an $\alpha < \kappa$ for which $f(x) \in U_{\alpha}$ and $f(y) \notin U_{\alpha}$, that is, $x \notin F_{\alpha}$ and $y \in F_{\alpha}$. Since $g^{-1}(g(F_{\alpha})) = F_{\alpha}$, it follows that $g(x) \notin g(F_{\alpha})$, while $g(y) \in g(F_{\alpha})$. Therefore, choosing an arbitrary point $z' \in g^{-1}(z)$ and setting h(z) = f(z') for every $z \in Z$, we obtain a well-defined map $h: Z \to X$. It is continuous, because

 $h^{-1}(U_{\alpha}) = M \setminus g(F_{\alpha})$ and $g(F_{\alpha})$ is closed by assumption. Clearly, $f = h \circ g$.

It remains to prove the implication $(3) \Rightarrow (1)$. Let φ be a continuous function $X \times D(\kappa) \to \mathbb{R}$. For each $\alpha \in D(\kappa)$, we define a function $f_{\alpha} \colon X \to \mathbb{R}$ by $f_{\alpha}(x) = \varphi(x, \alpha)$ for $x \in X$ and set $f = \Delta_{\alpha < \kappa} f_{\alpha} \colon X \to \mathbb{R}^{\kappa}$. Condition (3) implies the existence of a second-countable space M and continuous maps $g \colon X \to M$ and $h \colon M \to \mathbb{R}^{\kappa}$ such that $f = h \circ g$. Let $\pi_{\alpha} \colon \mathbb{R}^{\kappa} \to \mathbb{R}$ denote the projection onto the α th coordinate for $\alpha < \kappa$, and let

$$\psi \colon M \times D(\kappa) \to \mathbb{R}, \quad (z, \alpha) \mapsto \pi_{\alpha}(h(z)).$$

The function ψ is continuous, and $\varphi = \psi \circ (g \times \mathrm{id}_{D(\kappa)})$. Since M is second-countable, it follows that the product $M \times D(\kappa)$ is \mathbb{R} -factorizable [3, Theorem 3.2]. Hence there exist second-countable spaces M' and S and continuous maps $i \colon M \to M'$, $q \colon D(\kappa) \to S$, and $\nu \colon M' \times S \to \mathbb{R}$ for which $\psi = \nu \circ (i \times q)$. Let $\mu = \nu \circ (i \times \mathrm{id}_S)$. Then μ is a continuous map $M \times S \to \mathbb{R}$ and $\psi = \mu \circ (\mathrm{id}_M \times q)$. We have $\varphi = \psi \circ (g \times \mathrm{id}_{D(\kappa)}) = \mu \circ (\mathrm{id}_M \times q) \circ (g \times \mathrm{id}_{D(\kappa)}) = \mu \circ (g \times q)$.

Proposition 2.4. Let κ be a cardinal, and let X be a space with $w(X) \leq \kappa$. If the product $X \times D(\kappa)$ is \mathbb{R} -factorizable, then X is second-countable.

Proof. Proposition 2.3 implies the existence of a second-countable space M and continuous maps $g: X \to M$ and $h: M \to X$ for which $\mathrm{id}_X = h \circ g$. Obviously, the maps g and h are homeomorphisms. \square

Proposition 2.4 strengthens Theorem 3.1 of [3].

Proposition 2.5. If $X \times D(\omega_1)$ is \mathbb{R} -factorizable, then so is $X^{\omega} \times D(\omega_1)$.

To prove this proposition, we need a lemma. Given a map $f: X \to Y$, by $f^{\times \omega}$ we denote the product map $X^{\omega} \to Y^{\omega}$. Let us say that a subset F of X^{ω} is a *strong zero set* in X^{ω} if there exists a second-countable space M and a continuous function $f: X \to M$ such that

$$F = (f^{\times \omega})^{-1} (f^{\times \omega}(F))$$
 and $f^{\times \omega}(F)$ is closed in M^{ω} .

Lemma 2.1. If $X \times D(\omega_1)$ is \mathbb{R} -factorizable, then any closed subset of X^{ω} is a strong zero set.

Proof. Suppose that there exists a closed set F in X^{ω} which is not a strong zero set. To obtain a contradiction, we will recursively construct second-countable spaces M_{β} , continuous maps $f_{\beta} \colon X \to M_{\beta}$,

and strictly decreasing strong zero sets F_{β} for $\beta < \omega_1$ so that

$$F_{\beta} = (f_{\beta}^{\times \omega})^{-1} (\overline{f_{\beta}^{\times \omega}(F)})$$
 and hence $f_{\beta}^{\times \omega}(F_{\beta}) = \overline{f_{\beta}^{\times \omega}(F)}$ (*)

for each $\beta < \omega_1$. We define M_0 to be a singleton and f_0 to be the map $X \to M_0$. For F_0 we take X^{ω} . Suppose that $\alpha > 0$ and M_{β} , f_{β} , and F_{β} are defined for all $\beta < \alpha$. We set

$$f_{\alpha}^* = \underset{\beta < \alpha}{\Delta} f_{\beta} \colon X \to \prod_{\beta < \alpha} M_{\beta} \quad \text{and} \quad F_{\alpha}^* = (f_{\alpha}^{* \times \omega})^{-1} (\overline{f_{\alpha}^{* \times \omega}(F)}).$$

From (*) it follows that $F_{\alpha}^* \subset F_{\beta}$ for all $\beta < \alpha$. The set F_{α}^* satisfies condition (*) and hence $F_{\alpha}^* \neq F$, because F is not a strong zero set. Clearly, $F \subset F_{\alpha}^*$. Take

$$(x_n)_{n<\omega}\in F_\alpha^*\setminus F\subset X^\omega.$$

Since F is closed in X^{ω} , there exists an $N < \omega$ and cozero sets $U_i \subset X$, $i \leq N$, such that

$$(x_n)_{n<\omega}\in U_1\times U_2\times\cdots\times U_N\times X\times X\times X\times\cdots\subset X^\omega\setminus F.$$

For each $n \leq N$, let $g_n \colon X \to [0,1]$ be a continuous function such that $g_n(x_n) = 0$ and $g_n(X \setminus U_n) \subset \{1\}$. We set

$$M_{\alpha} = \mathbb{R}^{N} \times \prod_{\beta < \alpha} M_{\beta}, \quad f_{\alpha} = \left(\bigwedge_{n \le N} g_{n} \right) \Delta f_{\alpha}^{*}, \quad \text{and} \quad F_{\alpha} = (f_{\alpha}^{\times \omega})^{-1} \left(\overline{f_{\alpha}^{\times \omega}(F)} \right).$$

Note that $(x_n)_{n<\omega} \notin F_{\alpha}$. Indeed, for each $i \leq N$, the *i*th coordinate of $f_{\alpha}^{\times\omega}((x_n)_{n<\omega})$ is

$$\left(\left(\sum_{n\leq N}g_n\right)\Delta f_\alpha^*\right)(x_i)\in\mathbb{R}^{i-1}\times\{0\}\times\mathbb{R}^{N-i}\times\prod_{\beta<\alpha}M_\beta,$$

while for every $((y_n)_{n\in\omega})\in F$, there exists an $i\leq N$ for which $y_i\notin U_i$, so that the *i*th coordinate of $f_{\alpha}^{\times\omega}((y_n)_{n<\omega})$ is

$$\left(\left(\bigwedge_{n\leq N}g_n\right)\Delta f_\alpha^*\right)(y_i)\in\mathbb{R}^{i-1}\times\{1\}\times\mathbb{R}^{N-i}\times\prod_{\beta<\alpha}M_\beta,$$

whence

$$\overline{f_{\alpha}^{\times \omega}(F)} \subset \overline{\bigcup_{i \leq N} M_{\alpha}^{i-1} \times (\mathbb{R}^{i-1} \times \{1\} \times \mathbb{R}^{N-i} \times \prod_{\beta < \alpha} M_{\beta}) \times M_{\alpha} \times M_{\alpha} \times \dots}$$

$$= \bigcup_{i \leq N} M_{\alpha}^{i-1} \times (\mathbb{R}^{i-1} \times \{1\} \times \mathbb{R}^{N-i} \times \prod_{\beta < \alpha} M_{\beta}) \times M_{\alpha} \times M_{\alpha} \times \dots$$

Thus, $F_{\alpha} \subsetneq F_{\alpha}^* \subset F_{\beta}$, $\beta < \alpha$.

Having constructed M_{α} , f_{α} , and F_{α} for all $\alpha < \omega_1$, we set

$$Y = \prod_{\alpha < \omega_1} M_{\alpha}$$
 and $f = \bigwedge_{\alpha < \omega_1} f_{\alpha} \colon X \to Y$.

By Proposition 2.3(3) there exists a second-countable space M and continuous maps $g: X \to M$ and $h: M \to Y$ for which $f = h \circ g$. We have $f^{\times \omega} = h^{\times \omega} \circ q^{\times \omega}$. Let

$$\varphi \colon Y^{\omega} = \left(\prod_{\alpha < \omega_1} M_{\alpha}\right)^{\omega} \to \prod_{\alpha < \omega_1} M_{\alpha}^{\omega}$$

be the obvious homeomorphism permuting factors. Then

$$(\varphi \circ h^{\times \omega}) \circ g^{\times \omega} = \varphi \circ f^{\times \omega} = \underset{\alpha < \omega_1}{\Delta} f_{\alpha}^{\times \omega} \colon X^{\omega} \to \prod_{\alpha < \omega_1} M_{\alpha}^{\omega}.$$

Note that the sets

$$F'_{\beta} = \prod_{\gamma \leq \beta} f_{\gamma}^{\times \omega}(F_{\gamma}) \times \prod_{\beta < \gamma < \omega_1} M_{\gamma}^{\omega}, \quad \beta < \omega_1,$$

are closed in $\prod_{\gamma<\omega_1}M_{\gamma}^{\omega}$ and $F_{\beta}=\left(\Delta_{\alpha<\omega_1}f_{\alpha}^{\times\omega}\right)^{-1}(F_{\beta}')$. Since F_{β} strictly decrease, it follows that $(\varphi\circ h^{\times\omega})^{-1}(F_{\beta}')$, $\beta<\omega_1$, form a strictly decreasing sequence of closed sets in the second-countable space M^{ω} . The complements to these sets form an uncountable open cover of the complement to their intersection having no countable subcover, which cannot exist, because M is hereditarily Lindelöf. This contradiction proves that any closed subset of X^{ω} is a strong zero set.

Proof of Proposition 2.5. It suffices to show that X^{ω} satisfies condition (2) of Proposition 2.3. Let $\{F_{\alpha}: \alpha < \omega_1\}$ be a family of zero sets in X^{ω} . By virtue of Lemma 2.1, for each $\alpha < \omega_1$, there exists a second-countable space M_{α} and a continuous map $f_{\alpha}\colon X\to M_{\alpha}$ for which

$$F_{\alpha} = (f_{\alpha}^{\times \omega})^{-1} (f_{\alpha}^{\times \omega}(F_{\alpha}))$$
 and $f_{\alpha}^{\times \omega}(F_{\alpha})$ is closed in M_{α}^{ω} . (**)

Let Y, f, M, g, h, and φ be defined as in the proof of Lemma 2.1. It follows from (**) that, for each $\beta < \omega_1$,

$$F_{\beta} = \left(\bigwedge_{\alpha < \omega_1} f_{\alpha}^{\times \omega} \right)^{-1} \left(\bigwedge_{\alpha < \omega_1} f_{\alpha}^{\times \omega}(F_{\beta}) \right) \quad \text{and} \quad \bigwedge_{\alpha < \omega_1} f_{\alpha}^{\times \omega}(F_{\beta}) \text{ is closed in } \prod_{\alpha < \omega_1} M_{\alpha}^{\omega},$$

whence

$$F_{\beta} = (\varphi \circ f^{\times \omega})^{-1} ((\varphi \circ f^{\times \omega})(F_{\beta}))$$
 and $F_{\beta} = (g^{\times \omega})^{-1} (g^{\times \omega}(F_{\beta})),$

because $\Delta_{\alpha < \omega_1} f_{\alpha}^{\times \omega} = \varphi \circ f^{\times \omega} = (\varphi \circ h^{\times \omega}) \circ g^{\times \omega}$. From the same considerations it follows that $g^{\times \omega}(F_{\beta}) = (\varphi \circ h^{\times \omega})^{-1} (\Delta_{\alpha < \omega_1} f_{\alpha}^{\times \omega}(F_{\beta}))$

and, therefore, $g^{\times\omega}(F_{\beta})$ is closed in X^{ω} . Thus, condition (2) of Proposition 2.3 does hold for $\{F_{\alpha}: \alpha < \omega_1\}$ (with M^{ω} and $g^{\times\omega}$ playing the roles of M and g).

Proposition 2.6. If $X \times D(\omega_1)$ is \mathbb{R} -factorizable, then X^{ω} is hereditarily Lindelöf and hereditarily separable.

Proof. In view of Proposition 2.5, it suffices to prove that X is hereditarily Lindelöf and hereditarily separable.

First, we show that X is hereditarily Lindelöf. Suppose it is not. Then there exists a right separated set in X, i.e., a subspace $R = \{x_{\alpha} : \alpha < \omega_1\} \subset X$ in which all initial segments $\{x_{\beta} : \beta < \alpha\}$, $\alpha < \omega_1$, are open [6]. For each $\beta < \omega_1$, let U_{β} be a cozero neighborhood of x_{β} in X such that $U_{\alpha} \cap R \subset \{x_{\gamma} : \gamma < \beta + 1\}$. Then the sets $F_{\alpha} = X \setminus \bigcup_{\beta < \alpha} U_{\beta}$, $\alpha < \omega_1$, are zero sets in X, and they strictly decrease, because $\alpha \in F_{\alpha} \setminus F_{\alpha+1}$ for each $\alpha < \omega_1$. Proposition 2.4 implies the existence of a second-countable space M and a continuous map $g : X \to M$ such that $F_{\alpha} = g^{-1}(g(F_{\alpha}))$ and $g(F_{\alpha})$ is closed in M for each $\alpha < \omega_1$. Therefore, $(g(F_{\alpha}))_{\alpha < \omega_1}$ is a strictly decreasing ω_1 -sequence of closed sets in the second-countable space M and $\{M \setminus g(F_{\alpha}) : \alpha < \omega_1\}$ is an uncountable open cover of $M \setminus \bigcup_{\alpha < \omega_1} g(F_{\alpha})$ containing no countable subcover, which cannot exist, because M is hereditarily Lindelöf.

Thus, X is hereditarily Lindelöf and, therefore, perfectly normal.

Let us prove that X is hereditarily separable. Suppose it is not. Then there exists a left separated set in X, i.e., a set $L = \{x_{\alpha} : \alpha < \omega_1\} \subset X$ such that $\{x_{\beta} : \beta < \alpha\}$ is closed in L for each $\alpha < \omega_1$ [6]. Clearly, the closed subsets $F_{\alpha} = \overline{\{x_{\beta} : \beta < \alpha\}}$ of X strictly increase. They are zero sets, because X is perfectly normal. According to Proposition 2.4, there exists a second-countable space M and a continuous map $g : X \to M$ such that $F_{\alpha} = g^{-1}(g(F_{\alpha}))$ and $g(F_{\alpha})$ is closed in M for each $\alpha < \omega_1$. Thus, $(g(F_{\alpha}))_{\alpha < \omega_1}$ is a strictly increasing ω_1 -sequence of closed subsets of M and $M' = \bigcup_{\alpha < \omega_1} g(F_{\alpha})$ is a nonseparable subspace of M, because any countable subset of M' is contained in $g(F_{\alpha})$ for some $\alpha < \omega$. Such an M' cannot exists, since M is hereditarily separable.

As is known, any space X whose square is hereditarily Lindelöf is submetrizable (see, e.g., [4, Lemma 8.2]). This implies the following assertion.

Corollary 2.1. Any space X for which $X \times D(\omega_1)$ is \mathbb{R} -factorizable is submetrizable.

Proposition 2.7. Any space X for which the product $X \times D(2^{\omega})$ is \mathbb{R} -factorizable is second-countable.

Proof. It follows from Proposition 2.6 that X is separable. Therefore, $w(X) \leq 2^{\omega}$ and X is second-countable by Proposition 2.4.

Corollary 2.2 (CH). Any space X for which the product $X \times D(\omega_1)$ is \mathbb{R} -factorizable is second-countable.

As mentioned in the introduction, the CH assumption cannot be omitted, because if $\mathfrak{b} > \omega_1$, then the product of the countable Fréchet–Urysohn fan and $D(\omega_1)$ is \mathbb{R} -factorizable.

Theorem 2.1. If a product $X \times Y$ of spaces is \mathbb{R} -factorizable and Y is not pseudo- \aleph_1 -compact, then

- (1) X^{ω} is hereditarily Lindelöf and hereditarily separable;
- (2) if $w(X) \leq \omega_1$, then X is second-countable;
- (3) under CH, X is second-countable.

Proof. It follows from Proposition 2.2 that $X \times D(\omega_1)$ is \mathbb{R} -factorizable. Hence Propositions 2.6 and 2.4 imply (1) and (2) and Corollary 2.2 implies (3).

3. R-Factorizable Products of Topological Groups

Theorem 3.1. For topological groups G and H, the following conditions are equivalent:

- (1) $G \times H$ is an \mathbb{R} -factorizable group;
- (2) the groups G and H are \mathbb{R} -factorizable and the product $G \times H$ is multiplicatively \mathbb{R} -factorizable.

Proof. First, we show that $(1) \Rightarrow (2)$. The projections of $G \times H$ onto the factors are open homomorphisms; hence by Theorem 1.2 the groups G and H are \mathbb{R} -factorizable. According to Theorem B, the product $G \times H$ is multiplicatively \mathbb{R} -factorizable.

Let us prove that $(2) \Rightarrow (1)$. The group G is ω -narrow, being \mathbb{R} -factorizable [1, Proposition 8.1.3]. According to a theorem of Guran (see [1, Theorem 3.4.23]), G is a topological subgroup of a product G' of second-countable topological groups. Since G is \mathbb{R} -factorizable, it follows that G is z-embedded in G' [1, Theorem 8.2.7]. Similarly, H is z-embedded in a product H' of second-countable topological groups. Hence $G \times H$ is z-embedded in $G' \times H'$ by Proposition 2.1. Since $G' \times H'$ is a product second-countable topological groups, it follows by Theorem 8.1.14 of [1] that $G' \times H'$ is an \mathbb{R} -factorizable group. According to [1, Theorem 8.2.7], the group $G \times H$ is \mathbb{R} -factorizable, because it is z-embedded in the \mathbb{R} -factorizable group $G' \times H'$.

Proposition 3.1. If a product $G \times H$ of topological groups is multiplicatively \mathbb{R} -factorizable, then either both groups G and H are ω -narrow or one of them is \mathbb{R} -factorizable.

Proof. Theorem 2.1 implies that either both groups G and H are pseudo- \aleph_1 -compact or one of them is Lindelöf. It remains to recall that any pseudo- \aleph_1 -compact group is ω -narrow [1, 3.4.31] and any Lindelöf group is \mathbb{R} -factorizable [1, 8.1.6].

Question 5. Is it true that if a product $G \times H$ of topological groups is multiplicatively \mathbb{R} -factorizable, then one of the groups G and H is \mathbb{R} -factorizable?

When the product $G \times H$ is a square, the answer to this question is positive even in the much more general case of topological universal algebras.

Recall that an n-ary operation on a set X is any map from X^n to X. A universal algebra is a nonempty set X together with a set of operations on X. If X is endowed with a topology and all of these operations are continuous, then X is called a topological universal algebra. The operations are indexed by the elements of a set Σ of symbols of operations. This set Σ , together with a map $\nu \colon \Sigma \to \omega$ assigning arity to each $\sigma \in \Sigma$, is called a signature. A universal algebra with a signature Σ is called a Σ -algebra. For $\sigma \in \Sigma$, the operation on a Σ -algebra with index σ is usually denoted by the same symbol σ . Groups are universal algebras with signature $\Sigma_{\rm gr} = \{e, ^{-1}, \cdot\}$, where e is the symbol of a nullary operation (which is identified with the identity element of a group), $^{-1}$ is the symbol of a unary operation (inversion), and \cdot is the symbol of a binary operation (multiplication). Topological groups are topological $\Sigma_{\rm gr}$ -algebras.

Let Σ be a signature. A map $\varphi \colon X \to Y$ of Σ -algebras is called a homomorphism if, for any $n \in \omega$, any symbol $\sigma \in \Sigma$ of an n-ary operation, and any $x_1, x_2, \ldots, x_n \in X$, $\varphi(\sigma(x_1, x_2, \ldots, x_n)) = \sigma(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_n))$.

Definition 3.1 ([9]). Let Σ be a signature. A topological Σ -algebra X is said to be \mathbb{R} -factorizable if, given any continuous function $f: X \to \mathbb{R}$, there exists a second-countable topological Σ -algebra Y, a continuous homomorphism $g: X \to Y$, and a continuous function $h: Y \to \mathbb{R}$ such that $f = h \circ g$.

Formally, Definition 3.1 as applied to a topological group treated as a universal algebra differs from the definition of the \mathbb{R} -factorizability of topological groups, because Definition 3.1 does not require Y to be a topological group, it only requires it to be a topological Σ_{gr} -algebra, that is, a set with a nullary operation e, a unary operation $^{-1}$, and a

binary operation \cdot on which no constraints (like associativity of multiplication and the familiar properties of e and $^{-1}$) are imposed. But in fact these definitions coincide, because it is easy to see that any homomorphic image of a group is a group and, therefore, Y is a topological group. Similar considerations apply to topological semigroups and paratopological groups.

The most studied and interesting case is that of universal algebras with finite signature. In [9] the general case was considered, in which the signature has arbitrary cardinality and is endowed with a topology. In what follows, we consider universal algebras with finite signature, in which case the signature is a finite discrete space.

In [9] the notion of an \mathbb{R} -factorizable product X^n over a signature Σ was introduced. In the case of a finite discrete signature, this notion coincides with that of a multiplicatively \mathbb{R} -factorizable product X^n . The following proposition follows from Theorem 12 in [9].

Proposition 3.2. Let X be a topological Σ -algebra with finite signature Σ such that the arities of all operations $\sigma \in \Sigma$ do not exceed n. If the product X^n is multiplicatively \mathbb{R} -factorizable, then X is an \mathbb{R} -factorizable topological Σ -algebra.

Corollary 3.1. If the square $G \times G$ of a topological group G is multiplicatively \mathbb{R} -factorizable, then the group G is \mathbb{R} -factorizable.

Theorem 3.2. Let G be a topological group. The product $G \times G$ is multiplicatively \mathbb{R} -factorizable if and only if the group $G \times G$ is \mathbb{R} -factorizable.

Proof. Suppose that the product $G \times G$ is multiplicatively \mathbb{R} -factorizable. Then the group G is \mathbb{R} -factorizable by Corollary 3.1. According to Theorem 3.1, the group $G \times G$ is \mathbb{R} -factorizable as well.

Conversely, if the group $G \times G$ is \mathbb{R} -factorizable, then the product $G \times G$ is multiplicatively \mathbb{R} -factorizable by Theorem B.

Corollary 3.2. Let G be a topological group homeomorphic to a dense subspace of a product X of Lindelöf Σ -groups. Then G^{κ} is \mathbb{R} -factorizable for any cardinal κ .

Proof. According to Theorem 8.4.6 of [1], any product of Lindelöf Σ -groups, in particular $X^{\kappa} \times X^{\kappa}$, is perfectly \varkappa -normal. Since $G^{\kappa} \times G^{\kappa}$ is homeomorphic to a dense subspace of $X^{\kappa} \times X^{\kappa}$, it follows by Theorem 5.1 of [2] that $G^{\kappa} \times G^{\kappa}$ is z-embedded in $X^{\kappa} \times X^{\kappa}$. By Theorem 3.2 $X^{\kappa} \times X^{\kappa}$ is multiplicatively \mathbb{R} -factorizable, because any product of Lindelöf Σ -groups is \mathbb{R} -factorizable [1, Theorem 8.1.14]. Therefore, so is $G^{\kappa} \times G^{\kappa}$. It remains to apply Corollary 3.1.

Proposition 3.2 implies also the following statements.

Corollary 3.3. Let G be a topological semigroup. If the product $G \times G$ is multiplicatively \mathbb{R} -factorizable, then the semigroup G is \mathbb{R} -factorizable.

Recall that a paratopological group is a group with a topology with respect to which multiplication is continuous. We say that a paratopological group is \mathbb{R} -factorizable if, given any continuous function $f:G\to\mathbb{R}$, there exists a second-countable paratopological group H, a continuous homomorphism $h:G\to H$, and a continuous function $g:H\to\mathbb{R}$ for which $f=g\circ h$.

Corollary 3.4. Let G be a paratopological group. If the product $G \times G$ is multiplicatively \mathbb{R} -factorizable, then G is \mathbb{R} -factorizable.

Proof. Let $f: G \to \mathbb{R}$ be a continuous function. Since G is a topological semigroup, by Corollary 3.3 there exists a second-countable topological semigroup H, a continuous homomorphism $h: G \to H$, and a continuous function $g: H \to \mathbb{R}$ for which $f = g \circ h$. We can assume that h is surjective. Since $h(x \cdot y) = h(x) \cdot h(y)$, it follows that, for any $y \in H$ and any $x \in h^{-1}(y)$, we have $h(x^{-1}) \cdot y = y \cdot h(x^{-1}) = h(e)$ and $h(e) \cdot y = y \cdot h(e) = y$. Therefore, H is a group with identity element h(e) and inversion $h(x)^{-1} = h(x^{-1})$ (recall that h is a surjection). Thus, H is a paratopological group and h is a group homomorphism. \square

Corollary 3.5. Let G be a paratopological group. If the product $G \times G$ is Lindelöf, then the paratopological group G is \mathbb{R} -factorizable.

Proof. By Proposition 2.1 (6) $G \times G$ is multiplicatively \mathbb{R} -factorizable, and by Corollary 3.4 G is \mathbb{R} -factorizable.

Corollary 3.6. If G is a topological group such that the group $G \times G$ is \mathbb{R} -factorizable and a topological group H is homeomorphic to G, then the group $H \times H$ is \mathbb{R} -factorizable.

Proof. If the product $G \times G$ is productively \mathbb{R} -factorizable, then the group $G \times G$ is \mathbb{R} -factorizable by Theorem 3.2. The group G is the image of $G \times G$ under the natural projection homomorphism, which is continuous and open. Therefore, G is \mathbb{R} -factorizable by Theorem 1.2.

4. Weird ℝ-Factorizable Groups

Theorems 2.1 and 3.1 have the following immediate consequence.

Theorem 4.1. If G is a weird \mathbb{R} -factorizable group, H is a topological group, and the group $G \times H$ is \mathbb{R} -factorizable, then

- (1) H^{ω} is hereditarily Lindelöf and hereditarily separable;
- (2) if $w(H) < \omega_1$, then H is second-countable;
- (3) under CH, H is second-countable.
- 4.1. Squares of \mathbb{R} -Factorizable Groups. Theorems 3.2 and 2.1 imply the following statement.

Theorem 4.2. If G is a topological group and the group $G \times G$ is \mathbb{R} -factorizable, then G is pseudo- \aleph_1 -compact.

Indeed, by Theorem 3.2 $G \times G$ is multiplicatively \mathbb{R} -factorizable and by Theorem 2.1 if G is not pseudo- \aleph_1 -compact, then it must be hereditarily Lindelöf, which is impossible.

Corollary 4.1. The square of a weird \mathbb{R} -factorizable group is never \mathbb{R} -factorizable.

4.2. R-Factorizable Groups of Uncountable Pseudocharacter.

Theorem 4.3. Any \mathbb{R} -factorizable group of uncountable pseudocharacter is pseudo- \aleph_1 -compact.

Corollary 4.2. Any weird \mathbb{R} -factorizable group has countable pseudocharacter, i.e., is submetrizable.

The proof of Theorem 4.3 is based on the following lemma.

Lemma 4.1. Suppose given a discrete family $\{U_{\alpha} : \alpha < \omega_1\}$ of cozero sets in a topological group G and nonempty zero sets $F_{\alpha} \subset U_{\alpha}$, $\alpha < \omega_1$. Then there exists a second-countable group H and a continuous homomorphism $h: G \to H$ such that $h(F_{\alpha})$ is closed and $h^{-1}(h(F_{\alpha})) = F_{\alpha}$ for each $\alpha < \omega_1$.

Proof. First, we choose sets $Q_n \subset \omega_1$, $n \in \omega$, so that, for every $\alpha \in \omega_1$,

$$\bigcap \{Q_n : \alpha \in Q_n\} = \{\alpha\}.$$

As such sets we can take any countable base of any second-countable topology on ω_1 . Let

$$S_n = \bigcup \{ F_\alpha : \alpha \in Q_n \}, \quad n \in \omega.$$

Since all F_{α} are pairwise disjoint, it follows that, for each $\alpha < \omega_1$, $F_{\alpha} \cap S_n \notin \emptyset$ if and only if $\alpha \in Q_n$, in which case $F_{\alpha} \subset S_n$. Moreover, if $\beta \neq \alpha$, then there exists an $n \in \omega$ such that $\alpha \in Q_n$ and $\beta \notin Q_n$, that is, $F_{\alpha} \subset S_n$ and $F_{\beta} \cap S_n = \emptyset$. Therefore,

$$F_{\alpha} = \bigcap \{ S_n : \alpha \in Q_n \}.$$

For each $\alpha < \omega_1$, we fix a continuous function $f_\alpha \colon G \to \mathbb{R}$ such that $f_{\alpha}^{-1}(\mathbb{R}\setminus\{0\})=U_{\alpha}$ and $f_{\alpha}^{-1}(1)=F_{\alpha}$. We set

$$\varphi_n = \sum_{\alpha \in Q_n} f_\alpha \colon G \to \mathbb{R}, \qquad n \in \omega.$$

Every function φ_n is well defined and continuous, because the family $\{U_{\alpha}: \alpha \in Q_n\}$ is discrete; hence there exists a second-countable group H_n , a continuous homomorphism $h_n: G \to H_n$, and a continuous function $g_n: H_n \to \mathbb{R}$ for which $\varphi_n = g_n \circ h_n$. Note that $S_n = \varphi_n^{-1}(\{1\}) = h_n^{-1}(g_n^{-1}(\{1\})).$

We set

$$h = \bigwedge_{n \in \omega} h_n \colon G \to \prod_{n \in \omega} H_n$$
 and $H = h(G)$.

Since $F_{\alpha} = \bigcap \{S_n : \alpha \in Q_n\}$, it follows that $x \in F_{\alpha}$ if and only if $h_n(x) \in g_n^{-1}(1)$ whenever $\alpha \in Q_n$. Thus,

$$F_{\alpha} = h^{-1}(\{(x_n)_{n \in \omega} \in H : x_n \in g_n^{-1}(1) \text{ for all } n \in \omega \text{ such that } \alpha \in Q_n\}).$$

It follows from the continuity and surjectivity of h that $h(F_{\alpha})$ is closed in H and $h^{-1}(h(F_{\alpha})) = F_{\alpha}$.

Proof of Theorem 4.3. Let G be an \mathbb{R} -factorizable group of uncountable pseudocharacter with identity element e. Suppose that G is not pseudo- \aleph_1 -compact and let $\{U_\alpha : \alpha < \omega_1\}$ be a discrete family of nonempty cozero sets. For each $\alpha < \omega_1$, we choose $x_{\alpha} \in U_{\alpha}$ and a zero set $Z_{\alpha} \subset U_{\alpha}$ so that

- (i) $e \in Z_{\alpha} \subset x_{\alpha}^{-1} \cdot U_{\alpha}$; (ii) $Z_{\alpha} \subset Z_{\beta}$ and $Z_{\alpha} \neq Z_{\beta}$ if $\beta < \alpha$.

This can easily be done by transfinite recursion as follows. For Z_0 we take any zero set containing e and contained in $x_0^{-1} \cdot U_0$. Assuming that $\alpha > 0$ and Z_{β} are defined for $\beta < \alpha$, we choose any zero set $F'_{\alpha} \subset U_{\alpha}$ containing x_{α}^{-1} and put $Z'_{\alpha} = (x_{\alpha}^{-1} \cdot F'_{\alpha}) \cap \bigcap_{\beta < \alpha} Z_{\beta}$. Since Z'_{α} is a G_{δ} set and the pseudocharacter of G is uncountable, it follows that there exists an $x \in Z'_{\alpha} \setminus \{e\}$; for Z_{α} we take the intersection of Z'_{α} with any zero set containing e and not containing x.

Let $F_{\alpha} = x_{\alpha} \cdot Z_{\alpha}$ for $\alpha < \omega_1$. Then the sets U_{α} and F_{α} satisfy the assumptions of Lemma 4.1. Hence there exists a second-countable group H and a continuous homomorphism $h: G \to H$ such that

(i) all $h(F_{\alpha})$ are closed and hence so are all

$$h(Z_{\alpha}) = h(x_{\alpha}^{-1} \cdot F_{\alpha}) = h(x_{\alpha}^{-1}) \cdot h(F_{\alpha});$$

(ii)
$$h^{-1}(h(F_{\alpha})) = F_{\alpha}$$
.

Let N be the kernel of h. Then $F_{\alpha} = N \cdot F_{\alpha}$ and

$$h^{-1}(h(Z_\alpha)) = h^{-1}(x_\alpha^{-1} \cdot F_\alpha) = N \cdot x_\alpha^{-1} \cdot F_\alpha = x_\alpha^{-1} \cdot F_\alpha = Z_\alpha.$$

Therefore, $h(Z_{\alpha}) \neq h(Z_{\beta})$ for $\alpha \neq \beta$.

Thus, the subsets $h(Z_{\alpha})$, $\alpha < \omega_1$, of H are closed and strictly decrease. Their complements form an uncountable open cover containing no countable subcover in the second-countable space $H \setminus \bigcap_{\alpha < \omega_1} h(Z_{\alpha})$. This contradiction shows that the family $\{U_{\alpha} : \alpha < \omega_1\}$ cannot exist and G is pseudo- \aleph_1 -compact.

Corollary 4.3. If G and H are topological groups at least one of which is of uncountable pseudocharacter and the group $G \times H$ is \mathbb{R} -factorizable, then both G and H are \mathbb{R} -factorizable and pseudo- \aleph_1 -compact.

Proof. According to Theorem 4.3, the group $G \times H$ is pseudo- \aleph_1 -compact; hence so are its images G and H under continuous open projection homomorphisms. By Theorem 1.2 they are also \mathbb{R} -factorizable.

It follows from Theorem 1.1 that every nonmetrizable compact group has uncountable pseudocharacter, because compact spaces do not admit strictly coarser Hausdorff topologies. Therefore, Corollary 4.3 implies the following theorem of [1].

Corollary 4.4 ([1, Theorem 8.5.11]). If G is a topological group, H is a nonmetrizable compact topological group, and the group $G \times H$ is \mathbb{R} -factorizable, then G is pseudo- \aleph_1 -compact.

Corollary 4.5. If an \mathbb{R} -factorizable group G contains a nonmetrizable compact subspace, then G is pseudo- \aleph_1 -compact.

Proof. Let K be a compact subspace of G. If G is not pseudo- \aleph_1 -compact, then by Corollary 4.2 it is submetrizable and hence so is K. Since compact spaces do not admit strictly coarser Hausdorff topologies, it follows that K is metrizable.

4.3. \mathbb{R} -Factorizable Groups of Regular Uncountable Weight. It follows from Theorems 8.5.2 and 8.5.8 of [1] that any \mathbb{R} -factorizable group G with $w(G)^{\omega} < 2^{\omega_1}$ is pseudo- \aleph_1 -compact. Therefore, under the assumption $2^{\omega} < 2^{\omega_1}$, any \mathbb{R} -factorizable group of weight ω_1 is pseudo- \aleph_1 -compact. In this section, we show that this assumption can be removed. Moreover, we prove that any \mathbb{R} -factorizable group of regular uncountable weight κ is pseudo- κ -compact.

We begin with a simple observation.

Remark 4.1. Any \mathbb{R} -factorizable group G of weight κ embeds in a product of κ -many second-countable groups as a subgroup.

Indeed, since G is Tychonoff, its topology has a base $\{B_{\alpha} : \alpha < \kappa\}$ consisting of cozero sets. Continuous functions $f_{\alpha} : G \to \mathbb{R}$ witnessing that the B_{α} are cozero separate points from closed sets, and each f_{α} factors through a continuous homomorphism $h_{\alpha} : G \to H_{\alpha}$ to a second-countable group H_{α} . Clearly, the homomorphisms h_{α} separate points from closed sets as well, so that the diagonal $\Delta_{\alpha < \kappa} h_{\alpha} : G \to \prod_{\alpha < \kappa} H_{\alpha}$ is a topological isomorphic embedding.

Theorem 4.4. Any \mathbb{R} -factorizable group of regular uncountable weight κ is pseudo- κ -compact.

Corollary 4.6. If G is a weird \mathbb{R} -factorizable group, then $w(G) > \omega_1$.

Proof of Theorem 4.4. Suppose that κ is a regular uncountable cardinal and an \mathbb{R} -factorizable group G of weight κ is not pseudo- κ -compact, i.e., contains a discrete family $\{U_{\alpha}: \alpha < \kappa\}$ of nonempty cozero sets. Take $y_{\alpha} \in U_{\alpha}$ for each $\alpha \in \kappa$ and let $Y = \{y_{\alpha}: \alpha < \kappa\}$. The set Y is C-embedded in G (by Remark 1.3) and G is z-embedded in a product $\prod_{\alpha < \kappa} H_{\alpha}$ of second-countable groups H_{α} (by Remark 4.1 and Theorem 8.2.7 of [1]); hence every set $P \subset Y$ is cozero in $\prod_{\alpha < \kappa} H_{\alpha}$ (because any such set is cozero in the discrete space Y).

Choose a countable base \mathscr{B}_{α} of the topology of H_{α} for each $\alpha < \kappa$. Recall that the standard base \mathscr{B} of the topology of $\prod_{\alpha < \kappa} H_{\alpha}$ consists of sets of the form $\prod_{\alpha < \kappa} U_{\alpha}$, where $U_{\alpha} = H_{\alpha}$ for all but finitely many $\alpha < \kappa$ and $U_{\alpha} \in \mathscr{B}_{\alpha}$ for the remaining $\alpha < \kappa$. Clearly, $|\mathscr{B}| \leq \kappa$; since $w(G) = \kappa$, it follows that $|\mathscr{B}| = \kappa$. Let us index the elements of \mathscr{B} by ordinals: $\mathscr{B} = \{B_{\alpha} : \alpha < \kappa\}$. For each $\alpha < \kappa$, we set $P_{\alpha} = B_{\alpha} \cap Y$.

Lemma 4.2. For any $M \subset Y$, there exists a countable set $C \subset \kappa$ such that $M = \bigcup_{\alpha \in C} P_{\alpha}$.

Proof. Let $M \subset Y$. There exists a continuous function $f: \prod_{\alpha < \kappa} H_{\alpha} \to \mathbb{R}$ such that $M = f^{-1}(\mathbb{R} \setminus \{0\}) \cap Y$. It is well known that any real-valued continuous function on a product of separable spaces depends on only countably many coordinates (see, e.g., [11]). This means that there exists a countable set $A \subset \kappa$ and a continuous function $g: \prod_{\alpha \in A} H_{\alpha} \to \mathbb{R}$ for which $f = g \circ \pi_A$ (we use the standard notation π_A for the projection $\prod_{\alpha \in \kappa} H_{\alpha} \to \prod_{\alpha \in A} H_{\alpha}$). Thus,

$$f^{-1}(\mathbb{R}\setminus\{0\}) = g^{-1}(\mathbb{R}\setminus\{0\}) \times \prod_{\alpha\in\kappa\setminus A} H_{\alpha}.$$

The open set $g^{-1}(\mathbb{R}\setminus\{0\})$ in the countable product $\prod_{\alpha\in A} H_{\alpha}$ is a countable union of elements of the standard base of this product. Clearly,

if U is any such element, then $U \times \prod_{\alpha \in \kappa \setminus A} H_{\alpha}$ is an element of the standard base for the product $\prod_{\alpha \in \kappa} H_{\alpha}$, i.e., $U = B_{\alpha}$ for some $\alpha < \kappa$. This immediately implies the required assertion.

In what follows, we identify Y with κ ; this can be done, e.g., by means of the bijection $y_{\alpha} \mapsto \alpha$.

Thus, if G is not pseudo- κ -compact, then there must exist sets $P_{\alpha} \subset \kappa$, $\alpha < \kappa$, such that any $A \subset \kappa$ is the union of fewer than κ of them. Our goal is to show that this is impossible.

Lemma 4.3. Suppose that sets $P_{\alpha} \subset \kappa$, $\alpha < \kappa$, are such that, for every $\alpha < \kappa$, there exist ordinals x and y and a set $M \subset [x, y]$ satisfying the following conditions:

- (i) $\alpha < x < y < \kappa$;
- (ii) for any $C \subset \alpha$,

$$M \neq \bigcup_{\beta \in C} P_{\beta} \cap [x, y].$$

Then there exists a set $M \subset \kappa$ which is not the union of fewer than κ sets P_{α} .

Proof. We recursively define ordinals $x_{\alpha}, y_{\alpha} < \kappa$ and sets $M_{\alpha} \subset [x_{\alpha}, y_{\alpha}]$ so that

- (i) $\beta < x_{\beta} < y_{\beta} < x_{\alpha} < y_{\alpha}$ whenever $\beta < \alpha < \kappa$;
- (ii) for any $C \subset \sup_{\beta < \alpha} x_{\beta}$ (in particular, for any $C \subset \alpha$),

$$M_{\alpha} \neq \bigcup_{\beta \in C} P_{\beta} \cap [x_{\alpha}, y_{\alpha}].$$

The set $M = \bigcup_{\alpha < \kappa} M_{\alpha}$ is as required. Indeed, suppose that $C \subset \kappa$, $|C| < \kappa$ and $M = \bigcup_{\beta \in C} P_{\beta}$. Then $C \subset \alpha$ for some $\alpha < \kappa$ (because κ is regular). Clearly, $M \cap [x_{\alpha}, y_{\alpha}] = M_{\alpha}$, whence $M_{\alpha} = \bigcup_{\beta \in C} P_{\beta} \cap [x_{\alpha}, y_{\alpha}]$. This contradiction proves what we need.

It remains to prove the existence of x, y and M satisfying the conditions in Lemma 4.3. Let $\alpha < \kappa$. If $\bigcup_{\beta < \alpha} P_{\beta} \subset \alpha + 1$, then we set $x = \alpha$, $y = \alpha + 2$, and $M = \{\alpha + 1\}$. Otherwise we set $A = \{\beta < \alpha : P_{\beta} \setminus (\alpha + 1) \neq \emptyset\}$, $\gamma = \sup_{\beta \in A} \min(P_{\beta} \setminus (\alpha + 1))$, $x = \alpha$, $y = \gamma + 2$, and $M = \gamma + 1$. For each $\beta < \alpha$, the intersection $C_{\beta} \cap [x, y]$ either is empty or contains an ordinal smaller than $\gamma + 1$; therefore, M cannot be represented as a union of such intersections. In view of Lemmas 4.2 and 4.3 the group G is pseudo- κ -compact.

References

- [1] A. Arhangel'skii, M. Tkachenko, Topological Groups and Related Structures, Atlantis Press/World Sci., Amsterdam–Paris, 2008.
- [2] R. L. Blair, Spaces in which special sets are z-embedded, Canad. J. Math. 28 (1976) 673–690.
- [3] R. L. Blair, A. W. Hager, z-Embedding in $\beta X \times \beta Y$, in: Set-Theoretic Topology, ed. by G. M. Reed, Academic Press, New York, 1977, pp. 47–72.
- [4] C. J. R. Borges, On stratifiable spaces, Pacif. J. Math. 17 (1) (1966) 1–16.
- [5] M. G. Charalambous, Dimension Theory: A Selection of Theorems and Counterexamples, Springer International, Cham, 2019.
- [6] I. Juhász, Cardinal Functions in Topology, Math. Centre Tracts 34, Amsterdam, 1971.
- [7] L. S. Pontryagin, Continuous Groups, Gos. Izd. Tekhn.-Teoret. Lit., Moscow, 1954.
- [8] E. Reznichenko, The product of Lindelöf groups and R-factorizability, Topol. Appl. 345 (2024) 108837.
- [9] E. Reznichenko, Extensions and factorizations of topological and semitopological universal algebras, Topol. Appl. (2025) 109256.
- [10] E. Reznichenko, O. Sipacheva, The free topological group on the Sorgenfrey line is not \mathbb{R} -factorizable, Topol. Appl. 160 (2013) 1184–1187.
- [11] K. A. Ross, A. H. Stone, Products of separable spaces, Amer. Math. Monthly, 71 (1964) 398–403.
- [12] O. Sipacheva, No product theorem for the covering dimension of topological groups, arXiv:2207.04961 (2022).
- [13] O. Sipacheva, A non- \mathbb{R} -factorizable product of \mathbb{R} -factorizable groups, arXiv:2303.08878 (2023).
- [14] O. Sipacheva, No subgroup theorem for the covering dimension of topological groups, arXiv:2303.04593 (2023).
- [15] M. G. Tkachenko, Factorization theorems for topological groups and their applications, Topol. Appl. 38 (1991) 21–37.

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