Topology and its Applications 373 (2025) 109529

Contents lists available at ScienceDirect

Topology and its Applications

journal homepage: www.elsevier.com/locate/topol

A non-$\mathbb{R}$-factorizable product of $\mathbb{R}$-factorizable groups ☆

Ol'ga Sipacheva

Lomonosov Moscow State University, Leninskie Gory 1, Moscow, 119991, Russia

ARTICLE INFO

Article history:
Received 7 April 2025
Received in revised form 15 July 2025
Accepted 16 July 2025
Available online 21 July 2025

MSC:
22A05
54F45

Keywords:
Topological group
$\mathbb{R}$-factorizable group
Product of $\mathbb{R}$-factorizable groups

ABSTRACT

An example of two zero-dimensional $\mathbb{R}$-factorizable groups whose product is not $\mathbb{R}$-factorizable is constructed. One of these groups is second-countable and the other Lindelöf to any finite power.

Definition 1 *([1]).* A topological group G is said to be $\mathbb{R}$-*factorizable* if, for every continuous function $f: G \to \mathbb{R}$, there exists a continuous homomorphism $h: G \to H$ to a second-countable topological group H and a continuous function $g: H \to \mathbb{R}$ such that $f = g \circ h$.

The study of $\mathbb{R}$-factorizable groups goes back to the work of Pontryagin, who proved the $\mathbb{R}$-factorizability of compact groups [2, Example 37] (see also [3, Theorem 8.1.1]), although the notion was explicitly introduced only as late as 1991 by Tkachenko in [1]. In the same paper Tkachenko asked whether or not the $\mathbb{R}$-factorizability of groups is preserved by finite products [1, Problem 4.1]; versions of this question (some of which still remain unanswered) can be found in [3].

The first examples of $\mathbb{R}$-factorizable groups G and H for which $G \times H$ is not $\mathbb{R}$-factorizable were given by this author [4] and, independently, Reznichenko [5]. All these examples were Lindelöf and had some additional properties (for example, Reznichenko constructed a pair of Lindelöf groups whose product was not pseudo-$\aleph_1$-compact and another pair of Lindelöf groups whose product was separable and contained a closed discrete subspace of cardinality 2^ω). In this paper, we construct two zero-dimensional $\mathbb{R}$-factorizable

☆ This work was financially supported by the Russian Science Foundation, grant 22-11-00075-P.
E-mail address: ovsipa@gmail.com.

https://doi.org/10.1016/j.topol.2025.109529

groups G_1 and G_2 such that G_2 is second-countable, G_1^n is Lindelöf for any positive integer n, and $G_1 \times G_2$ is not $\mathbb{R}$-factorizable, thereby solving Problems 8.5.2, 8.5.4, and 8.5.6 and one half of Problem 8.5.5 in [3] (the last problem is whether the product of an $\mathbb{R}$-factorizable group and (a subgroup of) a σ-compact group is $\mathbb{R}$-factorizable).

We use $\mathbb{R}$ for the set of real numbers, $\mathbb{N}$ for the set of positive integers, and ω for the set of nonnegative integers. By $\oplus$ we denote the topological sum of spaces and by $|A|$, the cardinality of a set A. The definitions of the covering dimensions dim and $\dim_0$ can be found in [6]. A topological space X is *zero-dimensional* if it has a base consisting of clopen sets and *strongly zero-dimensional* if any finite cover of X by cozero sets has a disjoint finite refinement (that is, $\dim_0(X) = 0$). A subset Y of a space X is said to be *C-embedded* in X if any real-valued continuous function on Y has a continuous extension to X, and Y is *z-embedded* in X if every zero set of Y is the trace on Y of some zero set of X. A space X is *submetrizable* if its topology contains a metrizable one.

The main result of this paper is the following theorem.

Theorem. *There exist Boolean (and hence Abelian) Hausdorff topological groups G_1 and G_2 with the following properties:*

(i) *G_1 and G_2 are $\mathbb{R}$-factorizable and zero-dimensional;*
(ii) *G_1 is submetrizable, and G_1^n is Lindelöf for any $n \in \mathbb{N}$;*
(iii) *G_2 is second-countable;*
(iv) *$G_1 \times G_2$ is not $\mathbb{R}$-factorizable.*

Our construction of the groups G_1 and G_2 is based on Przymusiński's notion of n-cardinality [7] and on his construction of Lindelöf spaces X and Y such that $X \times Y$ is normal and $\dim X = \dim Y = 0$ but $\dim(X \times Y) > 0$ [8]. Below we recall some details, following the exposition of the construction given in [6].

Definition 2 *([7]).* Let X be a set, and let $n \in \mathbb{N}$. The *n-cardinality* (with respect to X) of a set $A \subset X^n$, denoted by $|A|_n$, is the least cardinal κ such that

$$A \subset \bigcup_{i=1}^{n} (X^{i-1} \times Y \times X^{n-i})$$

for some $Y \subset X$ with $|Y| = \kappa$ (here and in what follows it is assumed that $X^0 \times Y = Y \times X^0 = Y$). Clearly, $|A|_1 = |A|$ and $|A|_n \le |A|$. If $|A|_n \le \omega$, then A is said to be *n-countable*; otherwise, A is said to be *n-uncountable*.

For $x \in X^n$ and $i \le n$, we denote the ith coordinate of x by x_i and the set of all coordinates of x by $\tilde{x}$; in other words, we assume that $x = (x_1, \dots, x_n)$ and set $\tilde{x} = \{x_1, \dots, x_n\}$.

Lemma 1 *(see [6, Lemma 24.1]). Given a set X, a positive integer n, and an infinite cardinal κ, the following conditions on $A \subset X^n$ are equivalent:*

(a) $|A|_n \ge \kappa$;
(b) *A contains a subset B of cardinality κ such that $\tilde{p} \cap \tilde{q} = \varnothing$ whenever p and q are distinct points of B.*

Definition 3 *([6, p. 186]).* Suppose given $n \in \mathbb{N}$, a set X, and a topology τ on X^n. A set $B \subset X$ is said to be *weakly n-Bernstein with respect to τ* if $|A \cap B^n|_n = 2^\omega$ for every n-uncountable τ-closed set $A \subset X^n$.

Abusing notation, given a topology τ on X, we denote the product topology on X^n by τ^n. The proof of the following lemma is very similar to that of Theorem 24.3 in [6].

Lemma 2 *(see [6, Theorem 24.3 and Proposition 24.4]). Let (X,τ) be a space with separable completely metrizable topology τ, and let μ be a topology on X^2 with the following properties:*

(i) $\mu \supset \tau^2$;
(ii) X^2 *contains at most* 2^ω 2*-uncountable* μ*-closed sets;*
(iii) $|A|_2 \geq 2^\omega$ *for any* 2*-uncountable* μ*-closed set* $A \subset X^2$.

Then X has pairwise disjoint subsets $B_1, B_2, \dots$ such that every B_i is weakly 2-Bernstein with respect to μ and weakly n-Bernstein with respect to τ^n for all $n \in \mathbb{N}$.

Proof. Let us denote the family of all 2-uncountable μ-closed subsets of X^2 by $\mathscr{A}_{\mu,2}$ and the family of all n-uncountable τ^n-closed subsets of X^n, $n \in \mathbb{N}$, by $\mathscr{A}_{\tau,n}$. Note that $\mathscr{A}_{\mu,2} \supset \mathscr{A}_{\tau,2}$ (because $\tau^2 \subset \mu$), $|\mathscr{A}_{\tau,n}| \leq 2^\omega$ for $n \in \mathbb{N}$ (because (X^n, τ^n) is second-countable), and $|\mathscr{A}_{\mu,2}| \leq 2^\omega$ (by assumption (ii) of the lemma). We set

$$\mathscr{A} = \mathscr{A}_{\mu,2} \cup \bigcup_{n\in\mathbb{N}} \mathscr{A}_{\tau,n}$$

and index the elements of $\mathscr{A}$ by ordinals less than 2^ω as $\mathscr{A} = \{A_\alpha : \alpha < 2^\omega\}$ so that each element is assigned 2^ω indices. Let $\alpha < 2^\omega$. If $A_\alpha \in \mathscr{A}_{\mu,2}$, then we set $n(\alpha) = 2$; otherwise, we denote by $n(\alpha)$ the unique $n \in \mathbb{N}$ $(n \neq 2)$ for which $A_\alpha \in \mathscr{A}_{\tau,n}$. For all $\alpha \in 2^\omega$ and $i \in \mathbb{N}$, we recursively choose points $p(\alpha, i) \in A_\alpha$ so that $\tilde{p}(\alpha, i) \cap \tilde{p}(\beta, j) = \varnothing$ if $\alpha \neq \beta$ or $i \neq j$ in precisely the same way as in the proof of Theorem 24.3 of [6]; the only difference is that, in the case $n(\gamma) = 2$, we use our assumption (iii) and Lemma 1 to find a $B \subset A_\gamma$ such that $|B| = 2^\omega$ and $\tilde{p} \cap \tilde{q} = \varnothing$ for any distinct $p, q \in B$. After that, following [6, Theorem 24.3], we set

$$B_i = \bigcup\{\tilde{p}(\alpha, i) : \alpha < 2^\omega\}$$

for each $i \in \mathbb{N}$. Clearly, $B_i \cap B_j = \varnothing$ if $i \neq j$. For each $n \neq 2$, any n-uncountable τ^n-closed subset A of X^n equals A_α for 2^ω indices $\alpha \in 2^\omega$, and we have $p(\alpha, i) \in A \cap B_i^{n(\alpha)}$ and $n(\alpha) = n$ for each of these α and all $i \in \mathbb{N}$. Since $\tilde{p}(\alpha, i) \cap \tilde{p}(\beta, i) = \varnothing$ for $\alpha \neq \beta$, it follows that $|A \cap B_i^n|_n \geq 2^\omega$ by Lemma 1. Similarly, we have $|A \cap B_i^2|_2 \geq 2^\omega$ for any 2-uncountable μ-closed (and hence for any 2-uncountable τ^2-closed) subset A of X^2. □

Let C be the Cantor set in $[0, 1] \subset \mathbb{R}$, and let ε be the usual topology on C (induced by the Euclidean topology of $\mathbb{R}$). In [6] a special topology μ on C^2 was defined which satisfies conditions (i)–(iii) of Lemma 2 for C and ε playing the roles of X and τ (see Lemmas 27.2 and the proof of Lemma 27.3 in [6]). By Lemma 2 C contains pairwise disjoint sets $S_1, S_2, \dots$ which are weakly 2-Bernstein with respect to μ and weakly n-Bernstein with respect to ε^n for all $n \in \mathbb{N}$. Note that the set $S = C \setminus (S_1 \cup S_2)$ is weakly 2-Bernstein with respect to μ and weakly n-Bernstein with respect to ε^n for all $n \in \mathbb{N}$ too, because it contains the set S_3 with these properties. In [6, proof of Theorem 27.5], given any partition $\{S, S_1, S_2\}$ of C into subsets that are weakly 2-Bernstein with respect to μ, topologies τ_1 and τ_2 on C were constructed which satisfied, in particular, the following conditions for $i = 1, 2$ (see [6, pp. 210, 211]):

(1) $\tau_i \supset \varepsilon$;
(2) any τ_i-neighborhood of any point of S_i is an ε-neighborhood of this point;
(3) τ_i has a base consisting of ε-closed sets;

(4) $\dim(C, \tau_i) = \dim_0(C, \tau_i) = 0$;
(5) $\dim((C, \tau_1) \times (C, \tau_2)) = \dim_0((C, \tau_1) \times (C, \tau_2)) = 1$.

We fix topologies τ_1 and τ_2 on C with these properties and set $C_i = (C, \tau_i)$ for $i = 1, 2$. Note that it follows from (2) that the restriction of the topology τ_2 to S_2 coincides with the topology induced on S_2 by ε. In what follows, by S_2 we mean the set S_2 endowed with this topology, i.e., treat S_2 as a subspace of (C, ε); this is a separable metrizable space. In [6, Example 27.8] it was shown that

(6) $\dim_0(C_1 \times S_2) > 0$.

Lemma 3. *The spaces C_1^n are Lindelöf for all $n \in \mathbb{N}$.*

Proof. We argue by induction on n.

Let γ be a τ_1-open cover of C_1. For each $x \in C_1$, choose an element V_x of γ containing x. In view of (2), each point $s \in S_1$ has an ε-open neighborhood U_s contained in V_s. Let $U = \bigcup_{s \in S_1} U_s$. Since S_1 is weakly 1-Bernstein with respect to ε and $C_1 \setminus U$ is an ε-closed set disjoint from S_1, it follows that $C_1 \setminus U$ is 1-countable, that is, countable. Let $\{U_{s_k} : k \in \mathbb{N}\}$ be a countable subcover of the ε-open cover $\{U_s : s \in S_1\}$ of S_1. Then $\{V_{s_k} : k \in \mathbb{N}\} \cup \{V_x : x \in C_1 \setminus U\}$ is a countable subcover of γ.

Suppose that $n > 1$ and C_1^k is known to be Lindelöf for every $k < n$. Let γ be a τ_1^n-open cover of C_1^n. Again, for each $x \in C_1^n$, we choose an element V_x of γ containing x. In view of (2) each point $s \in S_1^n$ has an ε^n-open neighborhood U_s contained in V_s. Let $U = \bigcup_{s \in S_1^n} U_s$. Since S_1 is weakly n-Bernstein with respect to ε^n and $C_1^n \setminus U$ is an ε^n-closed set disjoint from S_1^n, it follows that $C_1^n \setminus U$ is n-countable, that is, there exists a countable set $Y \subset C_1$ such that

$$C_1^n \setminus U \subset \bigcup_{k=1}^{n} (C_1^{k-1} \times Y \times C_1^{n-k}).$$

This means that $C_1^n \setminus U$ is contained in the countable union of spaces of the form $C_1^{k-1} \times \{x\} \times C_1^{n-k}$, where $k \le n$ and $x \in Y$, each of which is homeomorphic to C_1^{n-1} and therefore Lindelöf by the induction hypothesis. It remains to choose a countable subfamily of γ covering $C_1^n \setminus U$ and a countable subfamily of $\{V_s : s \in S_1^n\}$ covering U, which exists because $\{V_s : s \in S_1^n\}$ has the ε^n-open refinement $\{U_s : s \in S_1^n\}$. □

Lemma 4. *Suppose that G_1, G_2, M_1, and M_2 are topological groups with the following properties:*

(i) *M_1 and M_2 are topological products of zero-dimensional second-countable topological groups;*
(ii) *G_1 and G_2 are subgroups of M_1 and M_2, respectively;*
(iii) *$C_1 \times S_2$ is C-embedded in $G_1 \times G_2$.*

Then the group $G_1 \times G_2$ is not $\mathbb{R}$-factorizable.

Proof. Any product of zero-dimensional second-countable topological spaces is strongly zero-dimensional [9]. Therefore, so is the product $M_1 \times M_2$, and it contains $G_1 \times G_2$ as a subgroup. As is known, any $\mathbb{R}$-factorizable subgroup of a topological group G is z-embedded in G [10, Theorem 3.2]. It follows that if the group $G_1 \times G_2$ were $\mathbb{R}$-factorizable, then this group, as well as its C-embedded subspace $C_1 \times S_2$, would be z-embedded in $M_1 \times M_2$. On the other hand, any z-embedded subspace of a strongly zero-dimensional space is strongly zero-dimensional [6, Theorem 11.22], while $\dim_0(C_1 \times S_2) > 0$. Hence $C_1 \times S_2$ is not z-embedded in $M_1 \times M_2$ and $G_1 \times G_2$ is not $\mathbb{R}$-factorizable. □

The product $C_1 \times S_2$ is surely C-embedded in $G_1 \times G_2$ when $C_1 \times S_2$ is a retract of $G_1 \times G_2$, which is the case if C_1 is a retract of G_1 and S_2 is a retract of G_2. Thus, we will look for topological groups G_1 and G_2 containing C_1 and S_2 as retracts. These G_1 and G_2 will be the Boolean groups $B(C_1)$ and $B(S_2)$, respectively, with special topologies.

A Boolean group is a group in which all elements are of order 2 (all such groups are Abelian), and the Boolean group $B(X)$ with basis X is the set $[X]^{<\omega}$ of finite subsets of X endowed with the operation $\triangle$ of symmetric difference. The zero element is the empty set. Each point $x \in X$ is identified with the singleton $\{x\}$. We use the notation $+_2$ for the group operation of $B(X)$ and occasionally write $\triangle$ instead of $+_2$. Thus, if $x \in X$, $F, G \in B(X)$, and $\mathbf{A} \subset B(X)$, then

$$x +_2 F = \{x\} +_2 F = \{x\} \triangle F, \qquad F +_2 G = F \triangle G,$$
$$F +_2 \mathbf{A} = \{F +_2 A : A \in \mathbf{A}\} = \{F \triangle A : A \in \mathbf{A}\}.$$

Let X be a topological space. The subgroups of $B(X)$ of the form

$$\mathbf{H}_\gamma = \{F \in B(X) : |F \cap U| \text{ is even for each } U \in \gamma\},$$

where γ ranges over all disjoint open covers of X, are normal (since $B(X)$ is Abelian), and the set of all these subgroups is obviously closed under the formation of finite intersections. Therefore, this set is a neighborhood base at zero of a group topology on $B(X)$ (see, e.g., [3, Theorem 1.3.12]). If X is zero-dimensional, then $B(X)$ with this topology contains X as a subspace, because given any γ and any $x \in X$, we obviously have $x +_2 \mathbf{H}_\gamma \cap X = U$, where U is the element of γ containing x (this element U is determined uniquely, because γ is disjoint). In what follows, we use the notation $B(X)$ for the abstract (that is, without topology) Boolean group with basis X and $B^{\mathrm{lin}}(X)$ for $B(X)$ with this topology.

Recall that a topological space is said to be *non-Archimedean* if it has a base $\mathscr{B}$ such that, for any $B_1, B_2 \in \mathscr{B}$, either $B_1 \cap B_2 = \varnothing$ or one of the sets B_1 and B_2 contains the other (see [12]). In Theorem 3 (version 2) of [13], for a non-Archimedean space X, a retraction of the subspace

$$B_{\mathrm{odd}}(X) = \{F \in B(X) : |F| \text{ is odd}\}$$

of $B^{\mathrm{lin}}(X)$ onto X was constructed (in [13] the group $B^{\mathrm{lin}}(X)$ was denoted by $B_z(X)$; our notation follows [11], where the groups $B^{\mathrm{lin}}(X)$ were studied in detail). In the particular case of the Cantor set C, the construction can be modified as follows.

Recall that C can be represented as the subset of $[0, 1]$ consisting of all numbers in $[0, 1]$ whose ternary expansions do not contain 1. This suggests the natural base $\mathscr{B}$ for the topology of C:

$$\mathscr{B} = \{U_{n_1 \dots n_k} : k \in \mathbb{N}, n_i \in \{0, 2\} \text{ for } i \le k\},$$

where $U_{n_1 \dots n_k}$ denotes the set of all numbers in $[0, 1]$ whose ternary expansions begin with $0.n_1 \dots n_k$. We also include the whole set C in $\mathscr{B}$. Clearly, the elements of $\mathscr{B}$ form a tree with respect to reverse inclusion and every element of $\mathscr{B}$ is clopen.

There are two natural orders on the set of subsets of C, the order by inclusion and the order induced by the usual order of $\mathbb{R}$. In what follows, when writing, say, "$A < B$," "$\min A$," or "A is on the left of B," we always mean the latter, unless otherwise is explicitly stated. Note that, given any two elements of $\mathscr{B}$, either one of them is contained in the other or one of them is on the left of the other.

Let F be any finite subset of C. We say that a set $A \subset C$ is *F-void* if $A \cap F = \varnothing$, *F-even* if $|A \cap F|$ is even and positive, and *A-odd* if $|A \cap F|$ is odd. Clearly, each F-even element of $\mathscr{B}$ is contained in a maximal (by inclusion) F-even element of $\mathscr{B}$, and the union of these maximal F-even elements is equal to the union of all F-even elements of $\mathscr{B}$. Moreover, this union is itself F-even, because $\mathscr{B}$ is a tree and therefore any two maximal F-even elements either coincide or are disjoint. Thus, no finite set $F \subset C$ of odd cardinality is covered by F-even elements of $\mathscr{B}$.

For $F \in B_{\mathrm{odd}}(C)$, we set

$$r(F) = \min(F \setminus \bigcup\{B \in \mathscr{B} : B \text{ is } F\text{-even}\}), \tag{$*$}$$

or, equivalently,

$$r(F) = \min(F \setminus \bigcup\{B \in \mathscr{B} : B \text{ is an inclusion-maximal } F\text{-even element of } \mathscr{B}\}).$$

Lemma 5. *There exists a second-countable zero-dimensional group topology τ on $B(C)$ such that it induces the Euclidean topology ε on C and the map $r\colon B_{\mathrm{odd}}(C) \to C$ defined by $(*)$ is continuous with respect to the topology $\tau|_{B_{\mathrm{odd}}(C)}$ (that is, τ restricted to $B_{\mathrm{odd}}(C)$).*

Proof. Recall that, given a disjoint open cover γ of C,

$$\mathbf{H}_\gamma = \{F \in B(C) : |F \cap U| \text{ is even for each } U \in \gamma\}.$$

The family

$$\mathscr{H} = \{\mathbf{H}_\gamma : \gamma \text{ is a disjoint cover of } C \text{ by elements of } \mathscr{B}\}$$

of subgroups of $B(C)$ is a neighborhood base at zero for a group topology τ of $B(C)$. This family is countable, because all open disjoint covers of C are finite (since C is compact) and $\mathscr{B}$ is countable. It is easy to check that C is contained in $(B(C), \tau)$ as a subspace. Indeed, take any point $x \in C$ and any neighborhood $V_x \in \mathscr{B}$ of x. Let γ be a disjoint cover of C consisting of V_x and some other elements of $\mathscr{B}$. If $F \in \mathbf{H}_\gamma$ and $x +_2 F = \{x\} \triangle F \in C$, then either $F = \varnothing$ or $F = \{x, y\}$. In the latter case, $x +_2 F = y$ and by the definition of $\mathbf{H}_\gamma$ the point y must belong to the same element of γ as x, that is, to V_x. Thus, $(x +_2 \mathbf{H}_\gamma) \cap C \subset V_x$. This shows that the topology induced by τ on C is not coarser than the topology ε of C. On the other hand, it cannot be finer, because τ is coarser than the topology of $B^{\mathrm{lin}}(C)$. Obviously, $(B(C), \tau)$ is T_0 and hence Tychonoff.

Note that all elements in any $\mathbf{H}_\gamma$ are of even cardinality. Therefore, for every $F \in B_{\mathrm{odd}}(C)$, we have $F +_2 \mathbf{H}_\gamma = \{F \triangle H : H \in \mathbf{H}_\gamma\} \subset B_{\mathrm{odd}}(C)$.

Let us show that the map r is continuous with respect to the topology $\tau|_{B_{\mathrm{odd}}(C)}$. Suppose that $x = r(F)$ for $F \in B_{\mathrm{odd}}(X)$. By construction $x \in F$. Take any neighborhood U of x. Let $V_1, \dots, V_m$ be all inclusion-maximal F-even elements of $\mathscr{B}$; their number is finite because they are pairwise disjoint (since $\mathscr{B}$ is a tree) and each of them intersects the finite set F. None of these elements contains x (because $x = r(F)$), and all of them are clopen. Choose a neighborhood $V_x \in \mathscr{B}$ of x satisfying the conditions $V_x \subset U$, $V_x \cap F = \{x\}$, and $V_x \cap V_i = \varnothing$ for $i \le m$. Consider the cover of C consisting of the sets V_x and $V_1, \dots, V_m$ and of all elements of $\mathscr{B}$ disjoint from them. This cover has a disjoint subcover γ, because any two of its elements are either disjoint or contained in one another (recall that $\mathscr{B}$ is a tree). Clearly, γ is finite. We claim that $r(F +_2 \mathbf{H}_\gamma) \subset V_x$.

Indeed, take an $H \in \mathbf{H}_\gamma$. We must show that $r(F \triangle H) \in V_x$. Note that an element V of γ is $(F \triangle H)$-odd if and only if it is F-odd, because each element of γ is either H-even or H-void and a point of F can be canceled in $F \triangle H$ only by some point of H. In particular, V_x is $(F \triangle H)$-odd.

Let V be the leftmost (with respect to the natural order $<$ on C) F-odd ($= (F \triangle H)$-odd) element of γ. Note that $V \cap F$ is disjoint from all inclusion-maximal F-even elements of $\mathscr{B}$, because all such elements are included in γ and V is not among them. By the definition of the map r we have $x = r(F) \le \min(V \cap F)$.

Since $x \in V_x$, it follows that V_x either coincides with V or is on the left of V, and since V_x is F-odd, it follows that $V = V_x$.

The point $r(F \vartriangle H)$ cannot belong to an $(F \vartriangle H)$-even or $(F \vartriangle H)$-void element of γ, because $\gamma \subset \mathscr{B}$ and

$$\begin{aligned} r(F \vartriangle H) &\in (F \vartriangle H) \setminus \bigcup\{B \in \mathscr{B} : B \text{ is } (F \vartriangle H)\text{-even}\} \\ &= (F \vartriangle H) \setminus \bigcup\{B \in \mathscr{B} : B \text{ is } (F \vartriangle H)\text{-even or } (F \vartriangle H)\text{-void}\}). \end{aligned}$$

Therefore, the element V of γ containing $r(F \vartriangle H)$ is $(F \vartriangle H)$-odd and hence either coincides with V_x or is on the right of V_x. Since $r(F \vartriangle H)$ is the least element of $F \vartriangle H$ not belonging to $\bigcup\{B \in \mathscr{B}\colon B$ is $(F \vartriangle H)$-even$\}$ and $r(F \vartriangle H) \in V$, it follows that there exists a family $\mathscr{B}'$ of $(F \vartriangle H)$-even elements of $\mathscr{B}$ such that

$$\bigcup \mathscr{B}' \supset \{y \in F \vartriangle H : y < V\}.$$

Suppose that $V \neq V_x$. Since V_x is $(F \vartriangle H)$-odd, we have $(F \vartriangle H) \cap V_x \neq \varnothing$. Let $W_1, \dots, W_k$ be all inclusion-maximal elements of $\mathscr{B}'$ intersecting $(F \vartriangle H) \cap V_x$. Each W_i, being an element of $\mathscr{B}$, either contains V_x or is contained in V_x, because $V_x \in \mathscr{B}$. By maximality the sets $W_1, \dots, W_k$ are pairwise disjoint. Therefore, if $k \geq 2$, then all of them are contained in V_x and $V_x \cap (F \vartriangle H) = \bigcup_{i \leq k} W_i \cap (F \vartriangle H)$. This is impossible, because $|V_x \cap (F \vartriangle H)|$ is odd and all $|W_i \cap (F \vartriangle H)|$ are even. Thus, some element W of $\mathscr{B}'$ contains $V_x \ni x$. Moreover, this W is a union of some elements of γ, since $V_x \in \gamma$, $W \in \mathscr{B}$, $\gamma \subset \mathscr{B}$, and γ covers C. This means that $|W \cap H|$ is even and therefore so is $|W \cap F|$, because W is $(F \vartriangle H)$-even. However, x equals $r(F)$ and hence does not belong to any F-even or F-void element of $\mathscr{B}$. This contradiction proves that $V = V_x$, i.e., $r(F \vartriangle H) \in V_x$.

Thus, $r(F +_2 \mathbf{H}_\gamma) \subset V_x$. We have shown that, for any $F \in B_{\text{odd}}(C)$ and any neighborhood U of $x = r(F)$ in C, there exists an $\mathbf{H}_\gamma \in \mathscr{H}$ such that the image of the open neighborhood $F +_2 \mathbf{H}_\gamma$ of F in $(B(C), \tau)$ under r is contained in U. This means that r is continuous with respect to the topology $\tau|_{B_{\text{odd}}(C)}$.

It remains to note that the group $(B(C), \tau)$ is zero-dimensional and metrizable, because the neighborhood base $\mathscr{H}$ at zero is countable and consists of open (and hence closed) subgroups, and it is separable, because

$$B(C) = \bigcup_{n \in \omega} B_n(C), \quad \text{where} \quad B_n(C) = \{F \in B(C) : |F| \leq n\},$$

and each $B_n(C)$ is the image of the separable space $(C \oplus \{\varnothing\})^n$ under the addition map $i_n\colon (x_1, \dots, x_n) \mapsto x_1 +_2 \dots +_2 x_n$, which is continuous with respect to any group topology on $B(C)$ inducing ε on C. □

Let $B^\tau(C)$ denote the group $B(C)$ with the topology τ defined in Lemma 5.

Lemma 6. *The Cantor space C is a retract of $B^\tau(C)$. Moreover, for any $x_0 \in C$, the map*

$$\hat{r}\colon B^\tau(C) \to C, \qquad \hat{r}(F) = \begin{cases} r(F) & \text{if } F \in B_{\text{odd}}(C), \\ x_0 & \text{otherwise}, \end{cases}$$

where r is defined by $(*)$, is a retraction.

Proof. The map $\hat{r}$ is continuous, because $B_{\text{odd}}(C)$ is clopen in $B^\tau(C)$, being a coset of the open subgroup

$$B_{\text{even}}(C) = \mathbf{H}_{\{C\}} = \{F \in B(C) : |F \cap C| = |F| \text{ is even}\}$$

of $B^\tau(C)$. Clearly, for every $x \in C$, we have

$$\hat{r}(x) = r(x) = \min(\{x\} \setminus \bigcup\{B \in \mathscr{B} : B \text{ is } \{x\}\text{-even}\}) = x,$$

because there are no $\{x\}$-even sets. Thus, $\hat{r}$ is a retraction. □

Now we can prove the main theorem.

Proof of the main theorem. We take the group $B^{\mathrm{lin}}(C_1)$ as G_1 and the subgroup of $B^\tau(C)$ generated by S_2 as G_2. According to [13, Theorem 7 (version 2)], C_1 is a retract of $B^{\mathrm{lin}}(C_1)$. Take any $x_0 \in S_2$. Restricting the retraction $\hat{r}$ defined in Lemma 6 for this x_0 to G_2, we obtain a retraction of G_2 onto S_2. Indeed, according to (∗), we have $r(F) \in F$ for any $F \in B(C)$. Therefore, $\hat{r}(F) \in F \cup \{x_0\} \subset S_2$ for any $F \in G_2$, whence $\hat{r}(G_2) = S_2$.

By Lemma 5 the group $B^\tau(C)$ is second-countable and zero-dimensional; hence so is its subgroup G_2. The topology of $B^{\mathrm{lin}}(C_1)$ is finer than τ, which implies the submetrizability of G_1. The same argument as at the end of the proof of Lemma 5 shows that G_1^n is Lindelöf for any $n \in \mathbb{N}$. In more detail,

$$B^{\mathrm{lin}}(C_1) = \bigcup_{n\in\omega} B_n(C), \quad \text{where} \quad B_n(C_1) = \{F \in B(C) : |F| \le n\},$$

and each $B_n(C_1)$ is the image of $(C_1 \oplus \{\varnothing\})^n$ under the continuous addition map $i_n\colon (x_1, \dots, x_n) \mapsto x_1 +_2 \dots +_2 x_n$. Hence $G_1 = B^{\mathrm{lin}}(C_1)$ is a continuous image of the sum $C_\infty = \bigoplus_{n\in\omega}(C_1 \oplus \{\varnothing\})^n$ and G_1^n is a continuous image of C_∞^n for every $n \in \mathbb{N}$. By Lemma 3 all spaces C_1^n are Lindelöf; therefore, so are $(C_1 \oplus \{\varnothing\})^n$ and C_∞^n. It follows that all G_1^n are Lindelöf.

Note that both groups G_1 and G_2 are $\mathbb{R}$-factorizable, being Lindelöf [1]. Let us show that $G_1 \times G_2$ is not. To this end, we first embed G_1 in a product of zero-dimensional second-countable groups and then apply Lemma 4.

Let Γ denote the set of all disjoint open covers of C_1. We fix a countable discrete space $D = \{d_n : n \in \mathbb{N}\}$ and denote by $B^{\mathrm{d}}(D)$ the Boolean group $B(D)$ endowed with the discrete topology. Note that any cover $\gamma \in \Gamma$ is countable, because C_1 is Lindelöf. Let $\gamma = \{U_n : n \in \mathbb{N}\}$ be such a cover. Consider the map $f_\gamma\colon C_1 \to D$ defined by $f_\gamma(U_n) = \{d_n\}$ for $n \in \mathbb{N}$. Let $\hat{f}_\gamma\colon G_1 \to B^{\mathrm{d}}(D)$ be the homomorphism extending f_γ to G_1; it is defined by $\hat{f}_\gamma(x_1 +_2 \dots +_2 x_n) = f_\gamma(x_1) +_2 \dots +_2 f_\gamma(x_1)$ for $x_1, \dots, x_n \in C_1$. The preimage $\hat{f}_\gamma^{-1}$ of the zero element $\varnothing$ of $B^{\mathrm{d}}(D)$ is precisely $\mathbf{H}_\gamma = \{F \in B^{\mathrm{lin}}(C_1) : |F \cap U| \text{ is even for each } U \in \gamma\}$; therefore, $\hat{f}_\gamma$ is continuous. Since the subgroups $\mathbf{H}_\gamma$, $\gamma \in \Gamma$, form a base of neighborhoods of zero for the topology of $B^{\mathrm{lin}}(X)$, it follows that the homomorphisms $\hat{f}_\gamma$, $\gamma \in \Gamma$, separate points from closed sets and therefore the diagonal

$$\mathop{\Delta}_{\gamma\in\Gamma} \hat{f}_\gamma\colon B^{\mathrm{lin}}(C_1) \to B^{\mathrm{d}}(D)^{|\Gamma|}$$

is a homeomorphic embedding; clearly, this is a homomorphism. Thus, $B^{\mathrm{lin}}(C_1)$ is topologically isomorphic to a subgroup of the power $B^{\mathrm{d}}(D)^{|\Gamma|}$ of the countable discrete group $B^{\mathrm{d}}(D)$, which gives us what we need.

Applying Lemma 4 to the groups G_1, G_2, $M_1 = B^{\mathrm{d}}(D)^{|\Gamma|}$, and $M_2 = G_2$, we see that $G_1 \times G_2$ is not $\mathbb{R}$-factorizable. □

The author is most grateful to Evgenii Reznichenko for very useful discussions.

References

[1] M.G. Tkachenko, Factorization theorems for topological groups and their applications, Topol. Appl. 38 (1991) 21–37.

[2] L.S. Pontryagin, Continuous Groups, Gos. Izd. Tekhn.-Teoret. Lit., Moscow, 1954.
[3] A. Arhangel'skii, M. Tkachenko, Topological Groups and Related Structures, Atlantis Press/World Sci., Amsterdam–Paris, 2008.
[4] O. Sipacheva, No subgroup theorem for the covering dimension of topological groups, arXiv:2303.04593 [math.GN].
[5] E. Reznichenko, The product of Lindelöf groups and $\mathbb{R}$-factorizability, Topol. Appl. 345 (2024) 108837.
[6] M.G. Charalambous, Dimension Theory: A Selection of Theorems and Counterexamples, Springer International, Cham, 2019.
[7] T.C. Przymusiński, On the notion of n-cardinality, Proc. Am. Math. Soc. 69 (1978) 333–338.
[8] T.C. Przymusiński, On the dimension of product spaces and an example of M. Wage, Proc. Am. Math. Soc. 76 (1979) 315–321.
[9] K. Morita, On the dimension of the product of Tychonoff spaces, Gen. Topol. Appl. 3 (1973) 123–133.
[10] S. Hernández, M. Sanchis, M.G. Tkachenko, Bounded sets in spaces and topological groups, Topol. Appl. 101 (2000) 21–43.
[11] O. Sipacheva, Free Boolean topological groups, Axioms 4 (2015) 492–517.
[12] P.J. Nyikos, Some surprising base properties in topology, in: N.M. Stavrakas, K.R. Allen (Eds.), Studies in Topology, Academic Press, New York–San Francisco–London, 1975, pp. 427–450.
[13] P.M. Gartside, E.A. Reznichenko, O.V. Sipacheva, Mal'tsev and retral spaces, Topol. Appl. 80 (1997) 115–129.