ARTICLE IN PRESS

Topology and its Applications ••• (••••) ••••••

Contents lists available at ScienceDirect

Topology and its Applications

journal homepage: www.elsevier.com/locate/topol

Discrete ultrafilters and homogeneity of product spaces

Anastasiya Groznova *, Ol'ga Sipacheva

Department of General Topology and Geometry, Faculty of Mechanics and Mathematics, M. V. Lomonosov Moscow State University, Leninskie Gory 1, Moscow, 199991 Russia

ARTICLE INFO

Article history:
Received 28 February 2022
Received in revised form 16 August 2022
Accepted 18 August 2022
Available online xxxx

MSC:
54H11
54G05
03E35

Keywords:
Discrete ultrafilter
Rudin–Frolik order
Rudin–Keisler order
Rudin–Blass order
$\beta\omega$-space
Homogeneous product space

ABSTRACT

An ultrafilter p on ω is said to be discrete if, given any function $f\colon \omega \to X$ to any completely regular Hausdorff space, there is an $A \in p$ such that $f(A)$ is discrete. Basic properties of discrete ultrafilters are studied. Three intermediate classes of spaces $\mathscr{R}_1 \subset \mathscr{R}_2 \subset \mathscr{R}_3$ between the class of F-spaces and the class of van Douwen's $\beta\omega$-spaces are introduced. It is proved that no product of infinite compact $\mathscr{R}_2$-spaces is homogeneous; moreover, under the assumption $\mathfrak{d} = \mathfrak{c}$, no product of $\beta\omega$-spaces is homogeneous.

In [1] Frolík proved the nonhomogeneity of the Stone–Čech remainder $\omega^* = \beta\omega \setminus \omega$ of ω by noticing that if two discrete sequences in ω^* converge to the same point $x \in \omega^*$ along ultrafilters p and q, then p and q are compatible in the Rudin–Frolík order. The idea is quite natural: if $p, q \in \omega^*$ are incompatible and $D = \{d_n : n \in \omega\}$ is a countable discrete subset of ω^*, then there cannot exist a homeomorphism $h\colon \omega^* \to \omega^*$ taking $q\text{-}\lim_n d_n$ to $p\text{-}\lim_n d_n$, because if it existed, then $(d_n)_{n\in\omega}$ and $(h(d_n))_{n\in\omega}$ would be discrete sequences converging to the same point $p\text{-}\lim_n d_n$ along p and q, respectively.

Frolík's idea of proving nonhomogeneity by considering orderings of ultrafilters was developed by Kunen. One of the key ingredients in his proof of the inhomogeneity of any product of infinite compact F-spaces is the following lemma on the Rudin–Keisler comparability of ultrafilters.

* Corresponding author.
E-mail addresses: avsav999@mail.ru (A. Groznova), osipa@gmail.com (O. Sipacheva).

https://doi.org/10.1016/j.topol.2022.108378
0166-8641/

Please cite this article in press as: A. Groznova, O. Sipacheva, Discrete ultrafilters and homogeneity of product spaces, Topol. Appl. (2022), https://doi.org/10.1016/j.topol.2022.108378

Kunen's lemma *([2, Lemma 4]). Let $p, q \in \omega^*$ be Rudin–Keisler incomparable weak P-points, and let X be any compact F-space. Suppose that $x \in X$, $(d_m)_{m\in\omega}$ is a discrete sequence of distinct points in X, $(e_n)_{n\in\omega}$ is any sequence of points in X, and $x = p\text{-}\lim_m d_m = q\text{-}\lim_n e_n$. Then $\{n : e_n = x\} \in q$.*

The present paper arose from an attempt to extend Kunen's lemma and, thereby, his result on the inhomogeneity of product spaces to other classes of spaces. This can be done by looking for larger classes for which Kunen's argument still works or by strengthening the assumptions on the ultrafilters p and q. In [3] we introduced new classes $\mathscr{R}_1$, $\mathscr{R}_2$, and $\mathscr{R}_3$ of topological spaces, which lie strictly between the classes of F- and $\beta\omega$-spaces, and proved that Kunen's lemma remains valid for $\mathscr{R}_2$-spaces. In this paper we mainly focus on special ultrafilters, namely, discrete ones, which are especially interesting in the context of homogeneity, because the convergence of any sequence along a discrete ultrafilter reduces to the convergence of a discrete subsequence.

1. Preliminaries

Throughout the paper by a space we mean a completely regular Hausdorff topological space.

Given a space X and $A \subset X$, by $\overline{A}$ we denote the closure of A in X and by $|A|$, the cardinality of A. Recall that sets $A, B \subset X$ are *separated* if $\overline{A} \cap B = A \cap \overline{B} = \varnothing$. A subspace $A \subset X$ is C^*-*embedded* in X if any continuous function $f\colon A \to [0, 1]$ has a continuous extension $\hat{f}\colon X \to [0, 1]$. We say that a sequence $(x_n)_{n\in\omega}$ of points of a space X is *discrete* if its range $\{x_n : n \in \omega\}$ is a discrete (not necessarily closed) subspace of X.

Every space X has *Stone–Čech compactification* βX; this is a compact space in which X is densely embedded so that any continuous map $f\colon X \to K$ to a Hausdorff compact space K has a continuous extension $\beta f\colon \beta X \to K$.

We use the standard notation $\mathbb{R}$ for the real line with the usual topology, $\mathbb{Q}$ for the space of rationals, ω for the set of nonnegative integers (endowed with the discrete topology when appropriate), $\beta\omega$ for the Stone–Čech compactification of ω, and ω^* for the Stone–Čech remainder $\beta\omega \setminus \omega$. It is well known that $\beta\omega$ is nothing but the space of ultrafilters on ω endowed with the topology generated by the base consisting of sets of the form $\overline{A} = \{p \in \beta\omega : A \in p\}$, where $A \subset \omega$; each $n \in \omega$ is identified with the principal ultrafilter $p(n) = \{A \subset \omega : n \in A\}$, so that ω^* is precisely the subspace of nonprincipal ultrafilters and ω is embedded in $\beta\omega$ as a dense open discrete subspace. Moreover, for each $A \subset \omega$, the set $\overline{A}$ defined above is indeed the closure of A in $\beta\omega$. Note also that an ultrafilter is nonprincipal if and only if all of its elements are infinite.

The *limit* of a sequence $(x_n)_{n\in\omega}$ in a space X *along an ultrafilter* p on ω, or the p-*limit* of $(x_n)_{n\in\omega}$, is a point $x \in X$, denoted by $p\text{-}\lim_n x_n$, such that, for any neighborhood U of x, the set $\{n \in \omega : x_n \in U\}$ belongs to p. We say that x is the *nontrivial* p-limit of $(x_n)_{n\in\omega}$ if $\{n \in \omega : x_n = x\} \notin p$. A sequence may have no p-limit, but if its p-limit exists, then it is unique (recall that we assume all spaces under consideration to be Hausdorff). Moreover, if X is compact, then any sequence $(x_n)_{n\in\omega}$ in X has precisely one p-limit for any $p \in \beta\omega$. To see this, it suffices to consider the continuous extension $\beta f\colon \beta\omega \to X$ of the map $f\colon \omega \to X$ defined by $f(n) = x_n$ for $n \in \omega$ and apply the following remark.

Remark 1. (i) Any ultrafilter $p \in \beta\omega$ is the p-limit of the sequence $(n)_{n\in\omega}$ in $\beta\omega$: $p = p\text{-}\lim_n n$.

(ii) Let $f\colon X \to Y$ be a continuous map of spaces X and Y, and let $x, x_n \in X$ for $n \in \omega$. If $p \in \beta\omega$ and $x = p\text{-}\lim_n x_n$, then $f(x) = p\text{-}\lim_n f(x_n)$.

(iii) Suppose that $p \in \omega^*$, $\varphi\colon \omega \to \omega$ is any function, $(x_n)_{n\in\omega}$ and $(y_k)_{k\in\omega}$ are two sequences in a space X, and there is a $P_0 \in p$ such that $y_{\varphi(n)} = x_n$ for $n \in P_0$. If $x = p\text{-}\lim_n x_n$, then $x = \beta\varphi(p)\text{-}\lim_k y_k$. Indeed, for any neighborhood U of x, we have $\{n \in \omega : x_n \in U\} \in p$. Therefore, $\{n \in P_0 : y_{\varphi(n)} \in U\} = P \in p$. Finally, $\{k \in \omega : y_k \in U\} \supset \varphi(P) \in \beta\varphi(p)$.

Please cite this article in press as: A. Groznova, O. Sipacheva, Discrete ultrafilters and homogeneity of product spaces, Topol. Appl. (2022), https://doi.org/10.1016/j.topol.2022.108378

Any function $f\colon \omega \to \omega$ can be treated as a map $\omega \to \beta\omega$ and, therefore, has a continuous extension $\beta f\colon \beta\omega \to \beta\omega$. This extension is explicitly described as

$$\beta f(p) = \{A \subset \omega : f^{-1}(A) \in p\}, \qquad p \in \beta\omega$$

(see [4, Lemma 3.30]). Ultrafilters p and q on ω are said to be *equivalent* if there exists a bijection $\varphi\colon \omega \to \omega$ such that $\beta\varphi(p) = q$.

Remark 2. Ultrafilters p and q on ω are equivalent if and only if there exists a function $f\colon \omega \to \omega$ and an $A \in p$ such that $\beta f(p) = q$ and the restriction $f \restriction A$ of f to A is one-to-one. Indeed, if such f and A exist, then there is a $B \subset A$, $B \in p$, for which $|\omega \setminus B| = |\omega \setminus f(B)| = \omega$, and any bijection $g\colon \omega \to \omega$ extending $f \restriction B$ witnesses the equivalence of p and q.

The extension βf of a function $f\colon \omega \to K$ to an arbitrary Hausdorff compact space K has a simple description as well: the image under βf of an ultrafilter p is defined by $\{\beta f(p)\} = \bigcap_{A\in p} \overline{f(A)}$ (see [4, Theorem 3.27]).

2. Orders on $\beta\omega$

There are several natural order relations on classes of equivalent ultrafilters; we will consider Rudin–Keisler, Rudin–Blass, and Rudin–Frolík orders. For detailed information about these and some other orders on $\beta\omega$, see, e.g., [4–6] and references therein.

The *Rudin–Keisler order* $\leqslant_{\mathrm{RK}}$ on $\beta\omega$ is defined by declaring that, for $p, q \in \beta\omega$, $p \leqslant_{\mathrm{RK}} q$ if and only if there exists a function $f\colon \omega \to \omega$ such that $\beta f(q) = p$.

The *Rudin–Blass order* $\leqslant_{\mathrm{RB}}$ on $\beta\omega$ is defined by declaring that, for $p, q \in \beta\omega$, $p \leqslant_{\mathrm{RB}} q$ if and only if there exists a finite-to-one function $f\colon \omega \to \omega$ such that $\beta f(q) = p$.

The *Rudin–Frolík order* $\leqslant_{\mathrm{RF}}$ on $\beta\omega$ is defined by declaring that, for $p, q \in \beta\omega$, $p \leqslant_{\mathrm{RF}} q$ if and only if there exists an injective function $\varphi\colon \omega \to \beta\omega$ such that $\varphi(\omega)$ is discrete and $\beta\varphi(p) = q$.

(Note that the map φ in the last definition and the maps f in the two preceding ones act on the ultrafilters in opposite directions.)

It is known that all these relations are indeed orders on the equivalence classes of ultrafilters. The relation $\leqslant_{\mathrm{RK}}$ is largest. Indeed, the implication $p \leqslant_{\mathrm{RB}} q \implies p \leqslant_{\mathrm{RK}} q$ is obvious, and if $p \leqslant_{\mathrm{RF}} q$, φ is the corresponding function in the definition of $\leqslant_{\mathrm{RF}}$, and A_i, $i \in \omega$, are disjoint subsets of ω determining disjoint neighborhoods $\overline{A}_i$ of the points $\varphi(i)$ in $\beta\omega$, then the function $f\colon \omega \to \omega$ defined by setting $f(n) = i$ if $n \in A_i$ and $f(n) = 0$ if $n \notin \bigcup A_i$ witnesses that $p \leqslant_{\mathrm{RK}} q$.

Remark 3. Each nonprincipal ultrafilter on ω has at most 2^ω $\leqslant_{\mathrm{RK}}$-predecessors (because the number of functions $\omega \to \omega$ is 2^ω).

Proposition 1. *For $p, q \in \beta\omega$, $p \leqslant_{\mathrm{RF}} q$ if and only if q is the p-limit of a discrete sequence $(x_n)_{n\in\omega}$ of distinct points in $\beta\omega$.*

Proof. According to Remark 1, we have $q = p\text{-}\lim_n x_n$ if and only if $q = \beta\varphi(p)$ for $\varphi\colon \omega \to \beta\omega$ defined by $\varphi(n) = x_n$ for $n \in \omega$. Therefore, if $\{x_n : n \in \omega\}$ is discrete and consists of distinct points, then $q = p\text{-}\lim_n x_n$ if and only if the map φ witnesses that $p \leqslant_{\mathrm{RF}} q$. □

The following theorem was proved by M. E. Rudin in [7] (see also [5, Theorem 16.16]).

Theorem 1 *([7]). The set of all $\leqslant_{\mathrm{RF}}$-predecessors of any ultrafilter $p \in \beta\omega$ is totally $\leqslant_{\mathrm{RF}}$-ordered.*

Please cite this article in press as: A. Groznova, O. Sipacheva, Discrete ultrafilters and homogeneity of product spaces, Topol. Appl. (2022), https://doi.org/10.1016/j.topol.2022.108378

Recall that a point x in a topological space X is a *P-point* if the intersection of any countable family of neighborhoods of x is also a neighborhood of x, x is a *weak P-point* if $x \notin \overline{D}$ whenever D is a countable subset of $X \setminus \{x\}$, and x is a *discretely weak P-point* if $x \notin \overline{D}$ whenever D is a countable discrete subset of $X \setminus \{x\}$. It is well known that $p \in \omega^*$ is a P-point in ω^* if, given any $f\colon \omega \to \omega$, there exists an $A \in p$ such that the restriction of f to A is either finite-to-one or constant, or, equivalently, given any sequence $(A_n)_{n\in\omega}$ of elements of p, there exists an $A \in p$ such that $A \subset^* A_n$ (i.e., $|A \setminus A_n| < \omega$) for each $n \in \omega$. The existence of weak P-points in ω^* can be proved in ZFC [8], while the existence of P-points in ω^* is independent of ZFC (see, e.g., [9]). Note that there exist discretely weak P-points in ω^* which are not weak P-points: an example of such a point is any lonely point in the sense of Simon, whose existence in ω^* was proved by Verner in [10]. We also mention that the non-discretely weak P-points in ω^* are precisely those of van Mill's type A_1 [11].

Related types of ultrafilters are selective and Q-point ultrafilters (see, e.g., [12], [13]). An ultrafilter $p \in \omega^*$ is said to be *selective*, or *Ramsey*, if, given any $f\colon \omega \to \omega$, there exists an $A \in p$ such that the restriction of f to A is either one-to-one or constant, or, equivalently, given any decreasing sequence $(A_n)_{n\in\omega}$ of elements of p, there exists an $A \in p$ such that $|A \cap (A_n \setminus A_{n+1})| \leqslant 1$ for each $n \in \omega$. An ultrafilter $p \in \omega^*$ is a *Q-point*, or *rare*, ultrafilter if, given any finite-to-one function $f\colon \omega \to \omega$, there exists an $A \in p$ such that the restriction of f to A is one-to-one.

The proof of the following theorem is essentially contained in [5].

Theorem 2. (i) *An ultrafilter $p \in \omega^*$ is minimal in the Rudin–Keisler order if and only if p is selective.*

(ii) *An ultrafilter $p \in \omega^*$ is minimal in the Rudin–Blass order if and only if p is a Q-point.*

(iii) *An ultrafilter $p \in \omega^*$ is minimal in the Rudin–Frolík order if and only if p is a discretely weak P-point in ω^*.*

Proof. Assertion (i) is Theorem 9.6 of [5]. Assertion (ii) (as well as (i)) easily follows from definitions and Remark 2. Assertion (iii) is Lemma 16.14 of [5]. □

Corollary 1. *If $p, q \in \omega^*$, $(x_n)_{n\in\omega}$ is a discrete sequence of distinct points in ω^*, and $q = p\text{-}\lim_n x_n$, then p is not a discretely weak P-point.*

Proof. By Proposition 1 $p \leqslant_{\mathrm{RF}} q$. If p is a discretely weak P-point, then p and q are equivalent by Theorem 2 (iii), so that q is a discretely weak P-point as well. But a discretely weak P-point cannot be limit for a discrete set. □

In the class of P-points $\leqslant_{\mathrm{RK}}$ coincides with $\leqslant_{\mathrm{RB}}$. Moreover, the following assertion holds.

Proposition 2. *If $p, q \in \omega^*$ and p is a P-point, then $q \leqslant_{\mathrm{RK}} p$ if and only if $q \leqslant_{\mathrm{RB}} p$.*

Proof. Only the 'only if' part needs to be proved. Suppose that $q \leqslant_{\mathrm{RK}} p$ and let $f\colon \omega \to \omega$ be a function for which $\beta f(p) = q$. Take $A \in p$ such that the restriction of f to A is finite-to-one and both sets $\omega \setminus A$ and $\omega \setminus f(A)$ are infinite. Let φ be any bijection between $\omega \setminus A$ and $\omega \setminus f(A)$. The function $g\colon \omega \to \omega$ coinciding with f on A and with φ on $\omega \setminus A$ is finite-to-one. Obviously, $\beta g(p) = q$. Therefore, $q \leqslant_{\mathrm{RB}} p$. □

The following amazing theorem of van Mill shows that the relation $\leqslant_{\mathrm{RF}}$ is very much smaller than $\leqslant_{\mathrm{RK}}$.

Theorem 3 *([6, Theorem 4.5.1]). There is a finite-to-one function $\pi\colon \omega \to \omega$ such that, given any $p \in \omega^*$, there is a weak P-point $q \in \omega^*$ for which $\beta\pi(q) = p$ (and hence $p \leqslant_{\mathrm{RB}} q$ and $p \leqslant_{\mathrm{RK}} q$).*

It is well known that there exist 2^{2^ω} $\leqslant_{\mathrm{RK}}$-incomparable (and hence $\leqslant_{\mathrm{RB}}$- and $\leqslant_{\mathrm{RF}}$-incomparable) ultrafilters in ω^* [14]. However, the problem of the existence of $\leqslant_{\mathrm{RK}}$-*incompatible* (i.e., having no common

Please cite this article in press as: A. Groznova, O. Sipacheva, Discrete ultrafilters and homogeneity of product spaces, Topol. Appl. (2022), https://doi.org/10.1016/j.topol.2022.108378

$\leqslant_{\mathrm{RK}}$-predecessor) ultrafilters is much more complicated. On the one hand, CH implies the existence of 2^{2^ω} nonequivalent $\leqslant_{\mathrm{RK}}$-minimal ultrafilters in ω^* [15, Sec. 8, Corollary 8]. Clearly, such ultrafilters cannot be compatible in any of the orders $\leqslant_{\mathrm{RK}}$, $\leqslant_{\mathrm{RB}}$ and $\leqslant_{\mathrm{RF}}$. On the other hand, it is consistent with ZFC that all ultrafilters in ω^* are *nearly coherent*, i.e., even $\leqslant_{\mathrm{RB}}$-compatible [16]. Finally, there exist (in ZFC) at least 2^ω nonequivalent (and even $\leqslant_{\mathrm{RK}}$-incomparable) weak P-points in ω^* [8], and such points are $\leqslant_{\mathrm{RF}}$-minimal and hence $\leqslant_{\mathrm{RF}}$-incompatible.

3. Discrete ultrafilters

In [17] Baumgartner introduced the notion of an I-ultrafilter and related classes of ultrafilters.

Definition 1 *([17])*. Let I be a family of subsets of a set X such that I contains all singletons and is closed under taking subsets. An ultrafilter p on ω is said to be an *I-ultrafilter* if, for any $f\colon \omega \to X$, there is an $A \in p$ such that $f(A) \in I$.

In the case where $X = \mathbb{R}$ and I is the family of all discrete (scattered, measure zero, nowhere dense) subsets of $\mathbb{R}$, an I-ultrafilter is said to be *discrete* (respectively, *scattered*, *measure zero*, *nowhere dense*).

Remark 4. Baumgartner also proved that *if $I = \{Y \subset 2^\omega : Y$ is finite or has order type of ω or $\omega + 1\}$, then the nonprincipal I-ultrafilters are exactly the P-points of ω^*.* (Here 2^ω is the Cantor set with the lexicographic order.) This immediately implies that *any P-point is discrete.*

Thus, we have

$$P\text{-point} \implies \text{discrete} \implies \text{scattered} \implies \text{measure zero} \implies \text{nowhere dense}.$$

Under Martin's axiom none of these implications reverses [17]. It makes no sense to speak about their reversibility without additional set-theoretic assumptions, because the nonexistence of nowhere dense ultrafilters is consistent with ZFC [9].

Considering families I of discrete subsets of other spaces X and imposing assumptions on $f\colon \omega \to X$, we obtain potentially different classes of discrete-like ultrafilters.

Definition 2. Let X be a space. We say that an ultrafilter p on ω is *X-discrete* (*finitely-to-one X-discrete*, *injectively X-discrete*) if, for any (respectively, for any finite-to-one, for any one-to-one) function $f\colon \omega \to X$, there is an $A \in p$ such that $f(A)$ is discrete in X. For $X = \mathbb{R}$, we write simply "discrete" instead of "$\mathbb{R}$-discrete."

Remark 5. Note that any ω^*-discrete (in any sense) ultrafilter p is $\beta\omega$-discrete (in the same sense). Indeed, take any $f\colon \omega \to \beta\omega$. If $A = f^{-1}(\omega) \in p$, then $f(A) \subset \omega \subset \beta\omega$ is discrete; otherwise $B = \omega \setminus A \in p$, and we can fix any distinct $q_n \in \omega^* \setminus f(B)$, $n \in \omega$, and consider the map $g\colon \omega \to \omega^*$ defined by $g(n) = f(n)$ for $n \in B$ and $g(n) = q_n$ for $n \in A$. Let $C \in p$ be such that $g(C)$ is discrete. Then $C \cap B \in p$ and $f(C \cap B) = g(C \cap B)$ is discrete.

Injectively discrete and injectively ω^*-discrete ultrafilters were considered in [18] (where they were called simply "discrete" and "ω^*-discrete"). The following proposition is similar to Proposition 12 of [18].

Proposition 3. *Every discrete (finitely-to-one discrete, injectively discrete) ultrafilter is X-discrete (respectively, finitely-to-one X-discrete, injectively X-discrete) for any space X.*

Proof. Let p be a discrete ultrafilter, and let $f\colon \omega \to X$ be any map. We set $x_n = f(n)$ for $n \in \omega$. For each pair (x_k, x_m) of different points in $f(\omega)$, take a continuous function $f_{(k,m)}\colon f(\omega) \to \{0, 1\}$ such that

Please cite this article in press as: A. Groznova, O. Sipacheva, Discrete ultrafilters and homogeneity of product spaces, Topol. Appl. (2022), https://doi.org/10.1016/j.topol.2022.108378

$f_{(k,m)}(x_k) = 0$ and $f_{(k,m)}(x_m) = 1$ (it exists because $f(\omega)$ is countable). The diagonal $g = \Delta\{f_{(k,m)} : x_k \neq x_m\}$ is a one-to-one continuous map of $f(\omega)$ to the Cantor set $2^\omega \subset \mathbb{R}$. Since p is discrete, it follows that there is an $A \in p$ for which $g(f(A))$ is discrete. Thus, $f(A)$ admits a one-to-one continuous map onto a discrete space; therefore, $f(A)$ is discrete. For finitely-to-one and injectively discrete ultrafilters, the proof is the same. □

Thus, any discrete ultrafilter is X-discrete for any space X, but it is unclear whether, say, an ω^*-discrete ultrafilter is discrete. The questions of whether injective discreteness implies finite-to-one discreteness and whether finite-to-one discreteness implies discreteness are not clear either, although the answer is unlikely to be positive. However, the nonexistence of injectively and finitely-to-one discrete ultrafilters, as well as that of discrete ones, in consistent with ZFC, because it follows from the nonexistence of nowhere dense ultrafilters. The following argument was kindly communicated to the authors by Taras Banakh.

Proposition 4. *The nonexistence of nowhere dense ultrafilters implies the nonexistence of injectively discrete ultrafilters.*

Proof. Let us say that an ultrafilter p on ω is injectively nowhere dense if, for any one-to-one function $f\colon \omega \to \mathbb{R}$, there is an $A \in p$ such that $f(A)$ is nowhere dense. We will prove that the nonexistence of nowhere dense ultrafilters implies that of injectively nowhere dense ultrafilters.

Suppose that there exist no nowhere dense ultrafilters but there exists and injectively nowhere dense ultrafilter p. Since p is not nowhere dense, it follows that we can find a function $f\colon \omega \to (0,1) \cong \mathbb{R}$ such that, for any $A \in p$, there exists an open set $U \subset (0,1)$ in which $U \cap f(A)$ is dense. Take an injective function $g\colon \omega \to \mathbb{R}$ such that $|g(n) - f(n)| < 2^{-n}$ for all n. Since p is injectively nowhere dense, there exists an $A \in p$ whose image $g(A)$ is nowhere dense in $\mathbb{R}$. On the other hand, the choice of f ensures that, for some open set U in R, the intersection $U \cap f(A)$ is dense in U. Thus, the set $U \cap f(A)$ has no isolated points. The condition $|g(n) - f(n)| \to 0$ implies that $U \cap g(A)$ is dense in U, and hence $g(A)$ cannot be nowhere dense in $\mathbb{R}$. □

Proposition 5. (i) *If $p \in \beta\omega$ is X-discrete for some space X and $q \leqslant_{\mathrm{RK}} p$, then q is X-discrete.*

(ii) *If $p \in \beta\omega$ is finitely-to-one X-discrete for some space X and $q \leqslant_{\mathrm{RB}} p$, then q is finitely-to-one X-discrete.*

(iii) *If $q \in \beta\omega$ is ω^*-discrete and p is the nontrivial q-limit of some sequence $(x_n)_{n\in\omega}$ in $\beta\omega$, then there exists an $r \in \beta\omega$ such that $r \leqslant_{\mathrm{RK}} q$ and $r \leqslant_{\mathrm{RF}} p$. Moreover, $p = r\text{-}\lim_n x_{k_n}$ for some discrete subsequence $(x_{k_n})_{n\in\omega}$ of $(x_n)_{n\in\omega}$ consisting of distinct points.*

(iv) *If $q \in \beta\omega$ is injectively ω^*-discrete, then $q \leqslant_{\mathrm{RF}} p$ if and only if p is the q-limit of some sequence $(x_n)_{n\in\omega}$ of distinct points of $\beta\omega$.*

Proof. The first two assertions are obvious, as well as the 'only if' part of the fourth one.

Let us prove (iii). Suppose that $(x_n)_{n\in\omega}$ is any sequence in $\beta\omega$ and p is the nontrivial q-limit of $(x_n)_{n\in\omega}$. If the ultrafilter q is principal, then there is nothing to prove, so we will assume that $q \in \omega^*$. Recall that, by Remark 5, the ω^*-discreteness of the ultrafilter q implies its $\beta\omega$-discreteness. Let $A \in q$ be such that the set $\{x_n : n \in A\}$ is discrete. This set is infinite, because p is the nontrivial q-limit of $(x_n)_{n\in\omega}$. Let $(x_{k_n})_{n\in\omega}$ be a subsequence of $(x_n)_{n\in\omega}$ consisting of distinct points and such that $\{x_n : n \in A\} = \{x_{k_n} : n \in \omega\}$. Note that $(x_{k_n})_{n\in\omega}$ is discrete.

Take any function $\pi\colon \omega \to \omega$ such that, for every $i \in A$, $\pi(i) = n$ if and only if $x_i = x_{k_n}$. The ultrafilter $r = \beta\pi(q)$ is a $\leqslant_{\mathrm{RK}}$-predecessor of q, and for each neighborhood U of p in $\beta\omega$, we have $\{n : x_{k_n} \in U\} = \pi(\{n : x_n \in U\}) \in r$. Therefore, $p = r\text{-}\lim_n x_{k_n}$. By Proposition 1 we have $r \leqslant_{\mathrm{RF}} p$.

Please cite this article in press as: A. Groznova, O. Sipacheva, Discrete ultrafilters and homogeneity of product spaces, Topol. Appl. (2022), https://doi.org/10.1016/j.topol.2022.108378

The proof of the 'if' part of assertion (iv) is similar, the only difference being that we must consider a one-to-one sequence $(x_n)_{n\in\omega}$; then the restriction of π to A is one-to-one, so that q is equivalent to r (see the argument in the proof of Theorem 2). $\square$

4. Products of ultrafilters

Recall that the *tensor*, or *Fubini*, *product* $p \otimes q$ of ultrafilters p and q on ω is the ultrafilter on $\omega \times \omega$ defined by

$$p \otimes q = \{A \subseteq \omega \times \omega : \{n : \{m : (n, m) \in A\} \in q\} \in p\}$$

(see, e.g., [4]). It is generated by a base consisting of sets of the form

$$\bigcup_{n\in P} \{n\} \times Q_n, \qquad \text{where } P \in p \text{ and } Q_n \in q \text{ for each } n \in P.$$

A generalization of the Fubini product of two ultrafilters is the *Fubini sum* $\sum_p(q_n)$ of a sequence of ultrafilters $q_n \in \beta\omega$ over $p \in \beta\omega$, which is generated by sets of the form

$$\bigcup_{n\in P} \{n\} \times Q_n, \qquad \text{where } P \in p \text{ and } Q_n \in q_n \text{ for each } n \in P.$$

Considering products of ultrafilters, we assume that $\omega \times \omega$ is endowed with the discrete topology, so that the space $\beta(\omega \times \omega)$ consists of ultrafilters on $\omega \times \omega$ and its topology is generated by the base of sets of the form $\overline{A} = \{p \in \beta(\omega \times \omega) : A \in p\}$. Thus, the spaces $\beta\omega$ and $\beta(\omega \times \omega)$ are homeomorphic and have the same description, and all notions and constructions related to $\beta\omega$ carry over to $\beta(\omega \times \omega)$ without any changes. In what follows, we identify ω with $\omega \times \omega$ and $\beta\omega$ with $\beta(\omega \times \omega)$ when appropriate.

In [11] van Mill defined various topological types of ultrafilters in ω^*, one of which was

$$A_1 = \{x \in \omega^* : \exists \text{countable discrete } D \subset \omega^* \setminus \{x\} \text{ with } x \in \overline{D}\}.$$

Thus, ultrafilters of van Mill's type A_1 are precisely those $p \in \omega^*$ which are not discretely weak P-points.

Proposition 6. *The tensor product $p \otimes q$ of any ultrafilters $p, q \in \omega^*$ belongs to A_1.*

Proof. Recall that, for $n \in \omega$, $p(n)$ denotes the principal ultrafilter on ω generated by $\{n\}$. Let $r = p\text{-}\lim_n(p(n) \otimes q)$. Any neighborhood of r in $\beta(\omega \times \omega)$ contains a neighborhood of the form $\overline{A}$ for $A \in r$. Since $r = p\text{-}\lim_n(p(n) \otimes q)$, for each $A \in r$, we have $B = \{n : A \in p(n) \otimes q\} \in p$. Hence, for every $n \in \omega$, there exists a $B_n \in q$ such that $\{n\} \times B_n \subset A$. Thus, each $A \in r$ contains $\bigcup_{n\in B} \{n\} \times B_n$ for some $B_n \in q$. It follows from the definition of the base of a product of ultrafilters that $r = p \otimes q$. Note that the set $D = \{p(n) \otimes q : n \in \omega\}$ is countable and discrete (disjoint neighborhoods of its elements are $\overline{\{n\} \times B_n}$) and $p \otimes q \in \overline{D}$, because any neighborhood of $p \otimes q$ contains a neighborhood of the form $\overline{A}$ for $A \in p \otimes q$, any A contains $B = \{n\} \times B_n$ for some $n \in \omega$ and $B_n \in q$, and $\overline{B}$ is a neighborhood of $p(n) \otimes q$ for any such $\overline{B}$. $\square$

Corollary 2. *A tensor product of two nonprincipal ultrafilters is never a discretely weak P-point. Thus, no such product is $\leqslant_{\mathrm{RF}}$-minimal.*

Proposition 7. *For any compact space X and any X-discrete (finitely-to-one X-discrete, injectively X-discrete) ultrafilters $p, q_n \in \omega^*$, $n \in \omega$, the Fubini sum $\sum_p(q_n)$ is an X-discrete (finitely-to-one X-discrete, injectively X-discrete) ultrafilter on $\omega \times \omega$.*

Please cite this article in press as: A. Groznova, O. Sipacheva, Discrete ultrafilters and homogeneity of product spaces, Topol. Appl. (2022), https://doi.org/10.1016/j.topol.2022.108378

Proof. Take any (any finite-to-one, any one-to-one) sequence $(x_{(n,m)}) \subset X$. We must show that there exists an $A \in \sum_p(q_n)$ for which the set $\{x_{(n,m)} : (n,m) \in A\}$ is discrete. For each $k \in \omega$, consider the sequence $(x_{(k,m)})_{m\in\omega}$. Since q_k is discrete, it follows that, for each $k \in \omega$, there exists a $B_k \in q_k$ for which the set $\{x_{(k,m)} : m \in B_k\}$ is discrete. Recall that, in a compact space, any sequence has a limit along any ultrafilter. Let $x_k = q_k\text{-}\lim_m x_{(k,m)}$ for $k \in \omega$. The X-discreteness of p implies the existence of a $C \in p$ for which the set $\{x_k : k \in C\}$ is discrete. Since $\{x_k : k \in C\}$ is countable, it is strongly discrete, that is, there exists a disjoint system of neighborhoods U_k of the points x_k in X (such a system is easy to construct by induction). For each neighborhood U_k, we have $\tilde{B}_k = \{m : x_{(k,m)} \in U_k\} \in q_k$, because $x_k = q_k\text{-}\lim_m x_{(k,m)}$. Thus, $A = \bigcup_{k\in C}(\{k\} \times (B_k \cap \tilde{B}_k)) \in \sum_p(q_k)$. Clearly, the set $\{x_{(n,m)} : (n,m) \in A\}$ is discrete. □

Corollary 3. *For any compact space X and any X-discrete (finitely-to-one X-discrete, injectively X-discrete) ultrafilters $p, q \in \omega^*$, the tensor product $p \otimes q$ is an X-discrete (finitely-to-one X-discrete, injectively X-discrete) ultrafilter on $\omega \times \omega$.*

Below we consider the set $\mathbb{N}$ of positive integers instead of ω solely in order that multiplication be a finite-to-one map. On the space $\beta\mathbb{N}$ the semigroup operations $\cdot$ and $+$ are defined (see [4]). Given two ultrafilters p and q on $\mathbb{N}$, their semigroup product $p \cdot q$ and sum $p + q$ are generated, respectively, by the sets

$$\bigcup_{n\in P}(n \cdot Q_n) \quad\text{and}\quad \bigcup_{n\in P}(n + Q_n), \quad \text{where } P \in p \text{ and } Q_n \in q \text{ for each } n \in P$$

(here $\cdot$ and $+$ denote the usual multiplication and addition in $\mathbb{N}$). The maps

$$\cdot\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N},\ (m,n) \mapsto m \cdot n, \quad\text{and}\quad +\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N},\ (m,n) \mapsto m + n,$$

are finite-to-one. Clearly, $\cdot(p \otimes q) = p \cdot q$ and $+(p \otimes q) = p + q$ for any $p, q \in \beta\mathbb{N}$. We obtain the following corollary.

Corollary 4. *For any compact space X and any X-discrete (finitely-to-one X-discrete) ultrafilters $p, q \in \mathbb{N}^*$, the ultrafilters $p \cdot q$ and $p + q$ are X-discrete (finitely-to-one X-discrete).*

Corollary 5. *The sets of discrete, finitely-to-one discrete, ω^*-discrete, and finitely-to-one ω^*-discrete ultrafilters on $\mathbb{N}$ are subsemigroups in the semigroups $(\beta\mathbb{N}, \cdot)$ and $(\beta\mathbb{N}, +)$.*

Corollary 6. *If there exist discrete ultrafilters, then there exist discrete ultrafilters which are not discretely weak P-points (and hence are not $\leqslant_{\mathrm{RF}}$-minimal).*

5. Classes of spaces between F-spaces and $\beta\omega$-spaces

In what follows, we consider extremally disconnected, F-, and $\beta\omega$-spaces. A space X is *extremally disconnected* if any disjoint open sets in X have disjoint closures (or, equivalently, if the closure of any open set is open), and X is called an *F-space* if any disjoint cozero sets in X are completely (= functionally) separated. Clearly, any extremally disconnected space is an F-space. The basic topological properties of extremally disconnected spaces and F-spaces can be found in the fundamental book [19] by Gillman and Jerison. The class of $\beta\omega$-spaces was introduced by van Douwen [20] as a generalization of the class of F-spaces. A space X is called a *$\beta\omega$-space* if, whenever D is a countable discrete subset of X with compact closure $\overline{D}$ in X, its closure $\overline{D}$ is the Stone–Čech compactification βD (or, equivalently, $\overline{D}$ is homeomorphic to $\beta\omega$).

It is known that any countable separated sets in an extremally disconnected space are functionally separated [21, 1.6]. It follows that any countable subspace of an extremally disconnected space is extremally

Please cite this article in press as: A. Groznova, O. Sipacheva, Discrete ultrafilters and homogeneity of product spaces, Topol. Appl. (2022), https://doi.org/10.1016/j.topol.2022.108378

disconnected. Moreover, any countable subspace of an F-space is extremally disconnected as well. Indeed, according to [19, 9H.1], any countable set in an F-space is C^*-embedded, and it is easy to see that the property of being an F-space is inherited by C^*-embedded subspaces. Thus, any countable subspace of an F-space is an F-space. It remains to note that all countable spaces are perfectly normal, so that any open set in such a space is a cozero set.

Note also that all countable subspaces of a space X are extremally disconnected if and only if any countable separated subsets of X have disjoint closures. Indeed, suppose that all countable subspaces of X are extremally disconnected and let A and B be countable separated subsets of X. Then $A \cup B$ is extremally disconnected, and A and B are separated in $A \cup B$. According to [21, Proposition 1.9], we have $\overline{A} \cap \overline{B} = \varnothing$. Conversely, suppose that countable separated subsets of X have disjoint closures and let Y be a countable subspace of X. Obviously, any disjoint open subsets of Y are separated in Y and hence in X. Therefore, they have disjoint closures in X and hence in Y.

These observations suggest a number of natural generalizations of the class of F-spaces.

Definition 3. We say that a topological space X is

- an $\mathscr{R}_1$-*space* if any countable subset of X is extremally disconnected, i.e., any two separated countable subsets of X have disjoint closures;
- an $\mathscr{R}_2$-*space* if any two separated countable subsets of X one of which is discrete have disjoint closures;
- an $\mathscr{R}_3$-*space* if any two separated countable discrete subsets of X have disjoint closures.

Importantly, the classes of $\mathscr{R}_i$-spaces are hereditary, unlike those of extremally disconnected and F-spaces. It is clear that

$$F\text{-spaces} \subset \mathscr{R}_1\text{-spaces} \subset \mathscr{R}_2\text{-spaces} \subset \mathscr{R}_3\text{-spaces}.$$

However, the reverse inclusions do not hold. Examples distinguishing between these classes are the quotient spaces $(\omega^* \oplus \omega^*)/\{p, q\}$, where p belongs to the first copy of ω^* and q, to the second one. An example of an $\mathscr{R}_1$-space which is not an F-space is obtained when both p and q are weak P-points not being P-points. An example of an $\mathscr{R}_2$-space which is not an $\mathscr{R}_1$-space is obtained when both p and q are discretely weak P-points not being weak P-points. Finally, an example of an $\mathscr{R}_3$-space which is not an $\mathscr{R}_2$-space is obtained when p is a discretely weak P-point not being a weak P-point and q is not a discretely weak P-point. For details, see [3].

Proposition 8. (i) *A space X is an $\mathscr{R}_3$-space if and only if any countable discrete set $D \subset X$ is C^*-embedded in $\overline{D}$.*

(ii) *A space X is an $\mathscr{R}_3$-space if and only if the closure of any countable discrete set $D \subset X$ in βX is the Stone–Čech compactification βD of D.*

(iii) *A space X is a $\beta\omega$-space if and only if any countable discrete set $D \subset X$ with compact closure is C^*-embedded in $\overline{D}$.*

Proof. (i) First, note that countable discrete sets $A, B \subset X$ are separated if and only if $D = A \cup B$ is discrete. Therefore, X is an $\mathscr{R}_3$-space if and only if any disjoint subsets of any discrete set $D \subset X$ have disjoint closures. By Taimanov's theorem (see [22, Theorem 3.2.1]) this means precisely that D is C^*-embedded in the closure of D in βX.

Assertions (ii) and (iii) immediately follow from (i) and the definitions of Stone–Čech compactification and of a $\beta\omega$-space. □

Corollary 7. *Any $\mathscr{R}_3$-space is a $\beta\omega$-space. Any compact $\beta\omega$-space is an $\mathscr{R}_3$-space.*

The class of $\beta\omega$-spaces is strictly larger than that of $\mathscr{R}_3$-spaces: any space containing no infinite compact subspaces is a $\beta\omega$-space but not necessarily a $\mathscr{R}_3$-space. From some point of view, the property of being an $\mathscr{R}_3$-space is more natural than that of being a $\beta\omega$-space.

A property which is in a sense opposite to $\mathscr{R}_3$ was introduced by Kunen in [2]. He called a space X *sequentially small* if any infinite set in X has an infinite subset whose closure does not contain a copy of $\beta\omega$. Thus, a compact space X is sequentially small if none of its countable discrete subsets is C^*-embedded.

6. Homogeneity in product spaces

In [3] we extended Kunen's lemma cited at the beginning of this paper as follows.

Proposition 9 *([3]). Let X be any compact $\mathscr{R}_2$-space. Suppose that $x \in X$, $(d_m)_{m\in\omega}$ is a discrete sequence of distinct points in X, $(e_n)_{n\in\omega}$ is any sequence of points in X, and $x = p\text{-}\lim_m d_m = q\text{-}\lim_n e_n$, where p is a weak P-point in ω^* and q is any point in ω^*. If $\{n : e_n = x\} \notin q$, then $p \leqslant_{\mathrm{RK}} q$.*

Imposing additional constraints on ultrafilters, we can further extend the class of spaces to which Kunen's lemma applies.

Proposition 10. *Let X be any compact $\beta\omega$-space. Suppose that $x \in X$, $(d_m)_{m\in\omega}$ is a discrete sequence of distinct points in X, $(e_n)_{n\in\omega}$ is any sequence of points in X, and $x = p\text{-}\lim_m d_m = q\text{-}\lim_n e_n$, where $p, q \in \omega^*$, p is a discretely weak P-point in ω^*, and q is discrete. If $\{n : e_n = x\} \notin q$, then $p \leqslant_{\mathrm{RK}} q$.*

Proof. Since q is discrete and $\{n : e_n = x\} \notin q$, it follows that there exists a $Q \in q$ for which the set $E = \{e_n : n \in Q\}$ is discrete (and $x \in \overline{E} \setminus E$). By assumption $D = \{d_m : m \in \omega\}$ is discrete as well, and $x \in \overline{D} \setminus D$. Since X is a compact $\beta\omega$-space, the point x has a neighborhood U such that either $U \cap (D \setminus \overline{E}) = \varnothing$ or $U \cap (E \setminus \overline{D}) = \varnothing$ (otherwise x would belong to the intersection of the closures of the separated countable discrete sets $E \setminus \overline{D}$ and $D \setminus \overline{E}$). By the definition of q-limit, the set $\{n \in \omega : e_n \in U\}$ belongs to q. We assume without loss of generality that Q is contained in this set.

Since $x \in \overline{D} \cap \overline{E}$ and $x \notin \overline{D \setminus \overline{E}}$ or $x \notin \overline{E \setminus \overline{D}}$, we have either $x \in \overline{D \cap \overline{E}}$ or $x \in \overline{E \cap \overline{D}}$.

Suppose that $x \in \overline{E \cap \overline{D}} \subset \overline{D}$. Clearly, we then have $U \cap (E \setminus \overline{D}) = \varnothing$. Recall that $\overline{D} = \beta D$ (because X is a compact $\beta\omega$-space) and consider the map $f\colon d_m \mapsto m$. We have $\beta f(x) = p\text{-}\lim_m \beta f(d_m) = p\text{-}\lim_m m = p$ (see Remark 1). On the other hand, setting $e'_n = e_n$ for $n \in Q$ and $e'_n = x$ for $n \in \omega \setminus Q$, we obtain a sequence $(e'_n)_{n\in\omega}$ for which $x = q\text{-}\lim_n e'_n$, because the sequence (e'_n) coincides with (e_n) on an element of q. Therefore, $p = \beta f(x) = q\text{-}\lim_n \beta f(e'_n)$ by Remark 1 (ii), and Proposition 5 (iii) implies $r \leqslant_{\mathrm{RK}} q$ for some $r \leqslant_{\mathrm{RF}} p$. Since p is a discretely weak P-point, we have $p = r$ by Theorem 2 (iii).

Now suppose that $x \in \overline{D \cap \overline{E}} \subset \overline{E}$; in this case, $U \cap (D \setminus \overline{E}) = \varnothing$, so that $\{m \in \omega : d_m \in \overline{E}\} \in p$. Let us somehow number the points of E as $\{e'_n : n \in \omega\}$, so that $(e'_n)_{n\in\omega}$ is a discrete sequence of distinct points with range E, and define $\varphi\colon \omega \to \omega$ by setting $\varphi(n)$ equal to, say, 0 for $n \in \omega \setminus Q$ and to the number k such that $e_n = e'_k$ for $n \in Q$. It is easy to check that $x = \beta\varphi(q)\text{-}\lim_n e'_n$ (see Remark 1 (iii)). Consider the one-to-one map $g\colon E \to \omega$ defined by $g(e'_n) = n$. We have $\overline{E} = \beta E$ and $\beta g(x) = \beta\varphi(q)\text{-}\lim_n \beta g(e'_n) = \beta\varphi(q)$.

Suppose that $\{m : d_m \notin E\} \in p$. Then there are $P, P' \in p$, $P \subset P'$, for which $\{d_m : m \in P'\} \subset E^* = \beta E \setminus E$ and $P' \setminus P$ is infinite. Choose a bijection $\Psi\colon P' \setminus P \to \omega \setminus P$. Setting $d'_m = d_m$ for $m \in P$ and $d'_m = d_{\Psi^{-1}(m)}$ for $m \in \omega \setminus P$, we obtain a new discrete sequence $(d'_m)_{m\in\omega}$ of distinct points with range $D' \subset D \cap E^*$ which coincides with $(d_m)_{m\in\omega}$ on $P \in p$. Clearly, we still have $x = p\text{-}\lim_m d'_m$ and $\beta g(x) = p\text{-}\lim_m \beta g(d'_m)$; moreover, $\beta g(d'_m) \in \omega^*$ for all m. But this is impossible by Corollary 1.

Thus, there exists a $P' \in p$ for which $\{d_m : m \in P'\} \subset E$. For the sequence $(d'_m)_{m\in\omega}$ constructed in precisely the same way as above (by taking $P \in p$, $P \subset P'$, such that $P' \setminus P$ is infinite and redefining (d_m) on $\omega \setminus P$), we have $D' = \{d'_m : m \in \omega\} \subset E$ and hence $(\beta g) \restriction D' = g \restriction D'$. Note also that $d'_m = d_m$ for $m \in P$.

Please cite this article in press as: A. Groznova, O. Sipacheva, Discrete ultrafilters and homogeneity of product spaces, Topol. Appl. (2022), https://doi.org/10.1016/j.topol.2022.108378

Since the element P of p has infinite complement in ω, there is a bijection $\psi\colon \omega \to \omega$ such that $\psi(g(d'_m)) = m$ for $m \in P$. Then the sequence $(m)_{m\in\omega}$, which coincides with $(\psi(g(d'_m)))_{m\in\omega}$ and hence with $(\psi(g(d_m)))_{m\in\omega}$ when restricted to P, converges to $\beta\psi(\beta g(x))$ along p. Therefore, $\beta\psi(\beta g(x)) = p$. Since ψ is one-to-one, it follows that p is equivalent to $\beta g(x)$, and since $\beta g(x) = \beta\varphi(q)$, it follows that $p \leqslant_{\mathrm{RK}} q$. □

Remark 6. For any ultrafilter $q \in \omega^*$, there exists a weak P-point $p \in \omega^*$ such that $p \nleqslant_{\mathrm{RK}} q$.

Indeed, by Remark 3 q has at most 2^ω $\leqslant_{\mathrm{RK}}$-predecessors, while the number of weak P-points in ω^* is 2^{ω^ω} [8].

Corollary 8. *If there exists a discrete ultrafilter in ω^*, then there exist no infinite homogeneous compact $\beta\omega$-spaces.*

Proof. Let q be a discrete ultrafilter in ω^*, and let $p \in \omega^*$ be a weak P-point such that $p \nleqslant_{\mathrm{RK}} q$. Suppose that X is an infinite homogeneous compact $\beta\omega$-space and $(e_n)_{n\in\omega}$ is any discrete sequence of distinct points in X. Let $x = p\text{-}\lim_n e_n$, and let $y = q\text{-}\lim_n e_n$. Since X is homogeneous, there exists a homeomorphism $h\colon X \to X$ taking y to x. By Remark 1, $x = q\text{-}\lim_n h(e_n)$, and by Proposition 10 we have $p \leqslant_{\mathrm{RK}} q$, which contradicts the assumption. □

Kunen used his lemma to prove a theorem on the nonhomogeneity of product spaces [2, Theorem 1]. Using Propositions 9 and 10 and Remark 6 instead of the lemma in Kunen's argument, we obtain the following results.

Theorem 4. *Let $X = \prod_{\alpha<\kappa} X_\alpha$, where κ is any cardinal and each X_α satisfies at least one of the following conditions:* (i) *is an infinite compact $\mathscr{R}_2$-space;* (ii) *contains a weak P-point;* (iii) *has a nonempty sequentially small open subset. Suppose also that at least one X_α is an infinite compact $\mathscr{R}_2$-space. Then X is not homogeneous.*

Corollary 9. *No product of compact $\mathscr{R}_2$-spaces is homogeneous.*

Theorem 5. *Suppose that there exists a discrete ultrafilter in ω^*. Let $X = \prod_{\alpha<\kappa} X_\alpha$, where κ is any cardinal and each X_α satisfies at least one of the following conditions:* (i) *is an infinite compact $\beta\omega$-space;* (ii) *contains a weak P-point;* (iii) *has a nonempty sequentially small open subset. Suppose also that at least one X_α is an infinite compact $\beta\omega$-space. Then X is not homogeneous.*

Corollary 10. *If there exists a discrete ultrafilter in ω^*, then no product of compact $\beta\omega$-spaces is homogeneous.*

The following corollary uses the assumption $\mathfrak{d} = \mathfrak{c}$. Recall that the notation $\mathfrak{d}$ is used for the *dominating number*, that is, the smallest cardinality of a family $\mathscr{D}$ of functions $\omega \to \omega$ with the property that, for every function $f\colon \omega \to \omega$, there is a $g \in \mathscr{D}$ such that $g(n) \geqslant f(n)$ for all but finitely many $n \in \omega$, and $\mathfrak{c}$ is the standard notation for 2^ω. Obviously, CH implies $\omega_1 = \mathfrak{d} = \mathfrak{c}$, although $\omega_1 < \mathfrak{d} = \mathfrak{c}$ is consistent with ZFC as well (see, e.g., [23]).

Corollary 11. *Under the assumption $\mathfrak{d} = \mathfrak{c}$, no product of compact $\beta\omega$-spaces is homogeneous.*

Proof. Ketonen proved that $\mathfrak{d} = \mathfrak{c}$ implies the existence of P-points in ω^* [24]. By Remark 4 any P-point is a discrete ultrafilter. □

In conclusion, we mention recent results of Reznichenko concerning homogeneous compact subspaces of product spaces. He proved that, under CH, (i) any compact set in a homogeneous subspace of a countable

Please cite this article in press as: A. Groznova, O. Sipacheva, Discrete ultrafilters and homogeneity of product spaces, Topol. Appl. (2022), https://doi.org/10.1016/j.topol.2022.108378

product of $\beta\omega$-spaces is metrizable, (ii) any compact set in a homogeneous subspace of a finite product of $\beta\omega$-spaces is finite [25, Theorems 3 and 4]. An analysis of his proof shows that CH can be replaced by the assumption that there exist uncountably many $\leqslant_{\mathrm{RB}}$-incompatible (that is, not nearly coherent) P-points. To be more precise, the following theorems hold.

Theorem 6. *Suppose that there exist uncountably many $\leqslant_{\mathrm{RB}}$-incompatible P-points and $X = \prod_{n\in\omega} X_n$, where each X_n is a compact $\beta\omega$-space. Let $Y \subset X$ be a homogeneous space. Then each compact subspace of Y is metrizable.*

Theorem 7. *Let n be a positive integer. Suppose that there exist $n+1$ $\leqslant_{\mathrm{RB}}$-incompatible P-points and $X = \prod_{i=1}^{n} X_i$, where each X_i is a compact $\beta\omega$-space. Let $Y \subset X$ be a homogeneous space. Then each compact subspace of Y is finite.*

This gives rise to the question of investigating conditions for the existence of uncountably many not nearly coherent P-points. A plausible conjecture is that such a condition is $\mathfrak{d} = \mathfrak{u} = \mathfrak{c}$ (here $\mathfrak{u}$ is the minimum cardinality of a free ultrafilter base on ω), because under this condition there exist, first, 2^{2^ω} Rudin–Keisler incomparable P-points [26] and, secondly, 2^{2^ω} near-coherence classes of ultrafilters [18].

Acknowledgements

The authors are very grateful to Evgenii Reznichenko and Taras Banakh for helpful discussions.

References

[1] Z. Frolík, Sums of ultrafilters, Bull. Am. Math. Soc. 73 (1967) 87–91.
[2] K. Kunen, Large homogeneous compact spaces, in: J. van Mill, G. Reed (Eds.), Open Problems in Topology, Elsevier Science Publishers B.V., North-Holland, Amsterdam, 1990, pp. 261–270.
[3] A.Yu. Groznova, O.V. Sipacheva, New properties of topological spaces related to extremal disconnectedness, Vestn. Mosk. Univ. Ser. 1. Mat. Mekh. (1) (2023), in press.
[4] N. Hindman, D. Strauss, Algebra in the Stone–Čech Compactification, Walter de Gruyter, Berlin, New York, 1998.
[5] W. Comfort, S. Negrepontis, The Theory of Ultrafilters, Springer-Verlag, Berlin, Heidelberg, New York, 1974.
[6] J. van Mill, An introduction to $\beta\omega$, Chapter 11, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, Elsevier Science Publishers B.V., North-Holland, Amsterdam, 1984, pp. 503–567.
[7] M.E. Rudin, Partial orders on the types in βN, Trans. Am. Math. Soc. 155 (2) (1971) 353–362.
[8] K. Kunen, Weak P-points in $\mathbb{N}^*$, Colloq. Math. Soc. János Bolyai 23 (1978) 741–749.
[9] S. Shelah, There may be no nowhere dense ultrafilter, in: Johann A. Makowsky, Elena V. Ravve (Eds.), Logic Colloquium '95: Proceedings of the Annual European Summer Meeting of the Association of Symbolic Logic, Haifa, Israel, August 9–18, 1995, in: Lecture Notes in Logic, vol. 11, Springer-Verlag, Berlin, Heidelberg, 1998.
[10] J.L. Verner, Lonely points revisited, Comment. Math. Univ. Carol. 54 (1) (2013) 105–110.
[11] J. van Mill, Sixteen topological types in $\beta\omega - \omega$, Topol. Appl. 13 (1982) 43–57.
[12] D. Booth, Ultrafilters on a countable set, Ann. Math. Log. 2 (1) (1970/71) 1–24.
[13] A.W. Miller, There are no Q-points in Laver's model for the Borel conjecture, Proc. Am. Math. Soc. 78 (1) (1980) 103–106.
[14] S. Shelah, M.E. Rudin, Unordered types of ultrafilters, Topol. Proc. 3 (1978) 199–204.
[15] A.R. Blass, Orderings of ultrafilters, Ph.D. thesis, Harvard Univ., Cambridge, Mass., 1970.
[16] A. Blass, S. Shelah, There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin–Keisler ordering may be downward directed, Ann. Pure Appl. Log. 33 (1987) 213–243.
[17] J.E. Baumgartner, Ultrafilters on ω, J. Symb. Log. 60 (1995) 624–639.
[18] T. Banakh, A. Blass, The number of near-coherence classes of ultrafilters is either finite or $2^{\mathfrak{c}}$, in: J. Bagaria, S. Todorcevic (Eds.), Set Theory, in: Trends in Mathematics, Birkhäuser, Basel, 2006, pp. 257–273.
[19] L. Gillman, M. Jerison, Rings of Continuous Functions, Springer, New York, 1960.
[20] E.K. van Douwen, Prime Mappings, Number of Factors and Binary Operations, Dissertationes Mathematicae, vol. 119, PWN, Warsaw, 1981.
[21] Z. Frolík, Maps of extremally disconnected spaces, theory of types, and applications, in: General Topology and Its Relations to Modern Analysis and Algebra: Proceedings of the Kanpur Topological Conference, 1968, Academia Publishing House of the Czechoslovak Academy of Sciences, Prague, 1971, pp. 133–142.
[22] R. Engelking, General Topology, Heldermann, Berlin, 1989.
[23] E.K. van Douwen, The integers and topology, Chapter 3, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, Elsevier Science Publishers B.V., North-Holland, Amsterdam, 1984, pp. 111–167.

Please cite this article in press as: A. Groznova, O. Sipacheva, Discrete ultrafilters and homogeneity of product spaces, Topol. Appl. (2022), https://doi.org/10.1016/j.topol.2022.108378

[24] J. Ketonen, On the existence of P-points in the Čech–Stone compactification of integers, Fundam. Math. 92 (1976) 91–94.
[25] E. Reznichenko, Homogeneous subspaces of products of extremally disconnected spaces, Topol. Appl. 284 (2020) 107403.
[26] N. Dobrinen, S. Todorcevic, Tukey types of ultrafilters, Ill. J. Math. 55 (3) (2011) 907–951.

Please cite this article in press as: A. Groznova, O. Sipacheva, Discrete ultrafilters and homogeneity of product spaces, Topol. Appl. (2022), https://doi.org/10.1016/j.topol.2022.108378