PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 149, Number 6, June 2021, Pages 2655–2668
https://doi.org/10.1090/proc/13992
Article electronically published on March 16, 2021

DISCRETE SUBSETS IN TOPOLOGICAL GROUPS AND COUNTABLE EXTREMALLY DISCONNECTED GROUPS

EVGENII REZNICHENKO AND OL'GA SIPACHEVA

(Communicated by Mirna Džamonja)

Abstract. In 1967 Arhangel'skii posed the problem of the existence in ZFC of a nondiscrete extremally disconnected topological group. The general case is still open, but we solve Arhangel'skii's problem for the class of countable groups. Namely, we prove that the existence of a countable nondiscrete extremally disconnected group implies the existence of a rapid ultrafilter; hence, such a group cannot be constructed in ZFC. We also prove that any countable topological group in which the filter of neighborhoods of the identity element is not rapid contains a discrete set with precisely one limit point, which gives a negative answer to Protasov's question on the existence in ZFC of a countable nondiscrete group in which all discrete subsets are closed.

Introduction and preliminaries

This work was motivated by the desire to solve the following problem of Arhangel'skii [1].

Problem (Arhangel'skii, 1967)**.** Does there exist in ZFC a nondiscrete Hausdorff extremally disconnected topological group?

The general case is still open, but in this paper we solve Arhangel'skii's problem for the class of countable groups. Namely, we prove that *the nonexistence of a countable nondiscrete Hausdorff extremally disconnected group is consistent with ZFC* (see Corollary 4.6). Since extremal disconnectedness is, obviously, inherited by dense subspaces, it follows that separable nondiscrete extremally disconnected groups cannot exist in ZFC either.

Recall that a topological space is said to be *extremally disconnected* if the closure of any open set in this space is open (or, equivalently, the closures of any two disjoint open sets are disjoint). Extremal disconnectedness is a classical notion of topology and functional analysis, and it plays a fundamental role in Boolean algebra. Extremally disconnected spaces were introduced by Stone [23] in order to characterize complete Boolean algebras (a Boolean algebra is complete if and only if its Stone space is extremally disconnected). Gleason proved that, in the category of compact spaces, the extremally disconnected spaces are precisely the projective objects [7], and Strauss extended his result to the category of regular Hausdorff spaces and perfect maps [24]. Moreover, each regular space X is the image of a uniquely determined extremally disconnected space $A(X)$ under an irreducible

Received by the editors December 23, 2016, and, in revised form, October 16, 2017.

2020 *Mathematics Subject Classification.* Primary 54G05, 54H11, 03E35, 22A05.

This work was supported by the Russian Foundation for Basic Research (project no. 15-01-05369).

perfect map π_X (the pair $(A(X), \pi_X)$ is called the projective resolution, or absolute, of X); see [26] for details. Finally, we mention the classical Nachbin–Goodner–Kelley theorem, which says that the injective objects in the category of Banach spaces and linear contractions are the spaces of continuous functions on extremally disconnected compact spaces [9].

It has long been known that an infinite extremally disconnected topological group cannot be compact; moreover, it cannot contain infinite compact sets [1]. However, Arhangel'skii's problem on the existence in ZFC of general (noncompact) extremally disconnected groups has not been solved so far. Still, some progress has been made. First, several consistent examples have been constructed [11–13, 22, 27, 28]. Most of these examples are countable, although Malykhin constructed (under various set-theoretic assumptions) a locally uncountable separable extremally disconnected group and a nondiscrete extremally disconnected group in which all countable subsets are closed and discrete [13]. Note that maximal topological groups (see definition in Section 4), which are an important special case of extremally disconnected groups, are always locally countable [12, 13]. The countable version of Arhangel'skii's problem was posed by various authors (see, e.g., [16, Problem 6] and [5, Question 6.1]): Does there exist a ZFC example of a countable nondiscrete Hausdorff extremally disconnected topological group? It has been proved that such an example cannot have maximal topology [18] (see also [30, Corollary 5.21]), and it cannot contain a nonclosed discrete set [29] or a sequence of countable open subgroups whose intersection has empty interior [21].

In this paper we solve the countable version of Arhangel'skii's problem by proving that the existence of a countable nondiscrete Hausdorff extremally disconnected topological group implies that of a rapid filter (recall that the nonexistence of rapid filters is consistent with ZFC [14]). Our solution is based on the following statement, which we regard as one of the two main results of this paper: *Any countable nondiscrete Hausdorff topological group whose identity element has nonrapid filter of neighborhoods contains a discrete subspace with precisely one limit point* (Theorem 2.6). Thus, nondiscrete Hausdorff countable topological groups in which all discrete subspaces are closed cannot exist in ZFC.

Thanks to Malykhin's beautiful theorem that any extremally disconnected topological group must contain an open Boolean subgroup (i.e., a subgroup consisting of elements of order 2) [12], in studying the existence of extremally disconnected groups, it suffices to consider only Boolean groups. Our second main result is that *if there are no rapid filters, then any countable nondiscrete Hausdorff Boolean topological group contains two disjoint discrete subsets for each of which the zero of the group is a unique limit point* (Theorem 3.3).

The paper is organized as follows. In the first section we introduce and study vast sets in groups, which are our main technical tool. In the second section we use them to construct nonclosed discrete sets in countable topological groups. The third section is devoted to nonclosed discrete sets in countable Boolean topological groups. In the last section we collect corollaries of the technical results of the first three sections, answer some known questions, and ask new questions.

A key role in our study is played by rapid filters on ω. They were introduced in [15] as filters whose elements form dominating families in ${}^\omega\omega$: a free filter $\mathscr{F}$ on ω is said to be *rapid* if every function $\omega \to \omega$ is majorized by the increasing enumeration of some element of $\mathscr{F}$. Clearly, any filter containing a rapid filter is rapid as well; thus, the existence of rapid filters is equivalent to that of rapid ultrafilters. Rapid ultrafilters are also known as semi-Q-point, or weak Q-point ultrafilters. In [14] Miller proved that the nonexistence of rapid (ultra)filters is consistent with ZFC and gave equivalent characterizations of rapid (ultra)filters; one of them, which is particularly convenient for our purposes, can be reformulated as follows: *A free filter $\mathscr{F}$ on ω is nonrapid if and only if, given any function $f\colon \omega \to \omega$, there exists a sequence $(T_n)_{n\in\omega}$ of finite subsets of ω such that each $F \in \mathscr{F}$ satisfies the condition $|F \cap T_n| \ge f(n)$ for some $n \in \omega$* (see [14, Theorem 3 (3)]).

We also mention Q-point, P-point, and selective ultrafilters on ω. A free ultrafilter $\mathscr{U}$ on ω is a *P-point*, or *weakly selective*, ultrafilter if, given any partition $\{A_n : n \in \omega\}$ of ω (or, equivalently, any increasing sequence $(A_n)_{n\in\omega}$ of subsets of ω) with $A_n \notin \mathscr{U}$, $n \in \omega$, there exists an $A \in \mathcal{U}$ such that $|A \cap A_n| < \aleph_0$ for all n. A free ultrafilter $\mathscr{U}$ on ω is said to be *Q-point*, or *rare*, if, given any partition $\{A_n : n \in \omega\}$ of ω into finite sets, there exists an $A \in \mathcal{U}$ such that $|A \cap A_n| = 1$ for all n. An ultrafilter which is simultaneously P-point and Q-point is said to be *selective*, or *Ramsey*. Any Q-point ultrafilter is rapid, but not vice versa (see, e.g., [14]). As mentioned above, the nonexistence of rapid (and, therefore, Q-point) ultrafilters is consistent with ZFC. The nonexistence of P-point ultrafilters is consistent as well (see [20]; Shelah's original proof is presented in [25]). However, it is still unknown whether the nonexistence of both rapid and P-point ultrafilters is consistent with ZFC.

Given a set X, we use $\mathrm{Ult}(X)$ to denote the set of ultrafilters on X and $\mathrm{Ult}^*(X)$, the set of free ultrafilters on X. For a topological space X and a point $x \in X$, by $\mathrm{Ult}_x(X)$ we denote the set of ultrafilters on X converging to x (i.e., containing all neighborhoods of x) and by $\mathrm{Ult}^*_x(X)$, the set of free ultrafilters on X converging to x. There is a natural topology on $\mathrm{Ult}(X)$, which turns this set into a compact extremally disconnected space, called the *ultrafilter space* of X (see, e.g., [4]); the set $\mathrm{Ult}^*(X)$, as well as $\mathrm{Ult}_x(X)$ and $\mathrm{Ult}^*_x(X)$ for any $x \in X$, is closed in $\mathrm{Ult}(X)$. If sets X and Y differ by finitely many elements, then $\mathrm{Ult}^*(X)$ coincides with $\mathrm{Ult}^*(Y)$. Each map $f\colon X \to Y$ induces the map $\mathrm{Ult}(f)\colon \mathrm{Ult}(X) \to \mathrm{Ult}(Y)$ defined by setting $\mathrm{Ult}(f)(\mathscr{U}) = \mathscr{V}$ if $f^{-1}(M) \in \mathscr{U}$ for each $M \in \mathscr{V}$.

Given $a < b < \omega$, we set $[a, b] = \{n \in \omega : a \le n \le b\}$. We use the standard notation $[A]^n$ for the set of all n-element subsets of a set A.

For simplicity, we assume all groups considered in this paper to be infinite and all topological groups, infinite and Hausdorff.

1. Vast sets

In this section we introduce vast sets in groups and describe their properties most important for our purposes.

Given a group G with identity element e and a positive integer m, let $\Phi_m(G)$ denote the family of all sets $M \subset G$ satisfying the following condition:

(Φ_m) for any $P \in [G]^m$, there exists a $Q \in [P]^2$ such that $Q^{-1}Q \subset M$;

this condition implies, in particular, that $e \in M$. We set $\Phi(G) = \bigcup_m \Phi_m(G)$.

Definition 1.1. Let G be a group with identity element e. We say that a set $M \subset G$ is *vast* if $M \in \Phi(G)$. Given a vast set M, we denote the minimum m for which $M \in \Phi_m(G)$ by J^G_M or simply J_M, when it is clear from the context which group G is meant.

First, we note that the intersections of vast sets with $P^{-1}P$ for large P are large.

Proposition 1.2. *Suppose that G is a group, $M \in \Phi(G)$, and n is a positive integer. Then there exists a positive integer m such that, for any $P \in [G]^m$, there is a $Q \in [P]^n$ for which $Q^{-1}Q \subset M$.*

Proof. Let $N = \max\{J_M, n\}$. By virtue of Ramsey's theorem [19], there exists a positive integer m such that any 2-edge-colored complete graph on m vertices contains a monochromatic clique on N vertices. Take $P \subset G$ with $|P| \geq m$. We set $P_0 = \{\{a, b\} \in [G]^2 : a^{-1}b \in M \cap M^{-1}\}$ and $P_1 = [P]^2 \setminus P_0$. There exists a $Q \subset P$ with $|Q| = N$ such that either $[Q]^2 \subset P_0$ or $[Q]^2 \subset P_1$. Since $N \geq J_M$, it follows that $[Q]^2 \cap P_0 \neq \varnothing$. Therefore, $[Q]^2 \subset P_0$ and $Q^{-1}Q \subset M$. □

Note also that the notion of vast sets is symmetric. The following proposition follows directly from the definition.

Proposition 1.3. *Suppose that G is a group and $M \in \Phi(G)$. Then*

(i) *$M \cap M^{-1} \in \Phi(G)$ and $J_M = J_{M\cap M^{-1}}$;*
(ii) *if $M \subset L$, then $L \in \Phi(G)$ and $J_M \geq J_L$;*
(iii) *$M^{-1} \in \Phi(G)$ and $J_M = J_{M^{-1}}$.*

Vast subsets of a group are large in a certain sense. We shall see below that vastness is organically related to another notion of largeness in semigroups, namely, syndeticity. This notion originated in topological dynamics in the context of the additive semigroup of positive integers. Below we define syndetic subsets of groups, although the term usually refers to semigroups; see [8] for details.

Definition 1.4 (see [8, Definition 4.38])**.** Let G be a group. A set $Q \subset G$ is *syndetic* if there exists a finite set $T \subset G$ such that $TQ = G$.

For $Q \subset G$, we set

$$I_Q = \min\{|T| : T \subset G \text{ and } TQ = G\};$$

Q is syndetic if and only if I_Q is finite.

Note that syndetic subgroups are precisely those of finite index, and totally bounded topological groups are precisely those in which all open sets are syndetic.

All vast sets are syndetic. To be more precise, the following assertion holds.

Proposition 1.5. *Suppose that G is a group with identity element e, $M \in \Phi(G)$, and $S \subset G$. Then there exist finite sets $Q, R \subset S$ with $|Q|, |R| < J_M$ such that $S \subset QM$ and $S \subset MR$. Moreover, M is syndetic and $I_M < J_M$.*

Proof. We can assume that $M = M^{-1}$. Let Q be a maximal subset of S for which $Q^{-1}Q \cap M \subset \{e\}$. Then $|Q| < J_M$ and, for any $s \in S \setminus Q$, there exists a $q \in Q$ such that $q^{-1}s \in M$ (because Q is maximal and $M = M^{-1}$). Hence $S \subset QM$. Repeating the same argument for S^{-1} instead of S, we see that there exists an $R \subset S$ with $|R| < J_M$ such that $S^{-1} \subset MR^{-1} = (RM)^{-1}$. Hence $S \subset RM$. To prove the second assertion, it suffices to take $S = G$. □

The converse is not true: there exist nonvast syndetic sets.

Example. Let G be a Boolean group with zero 0, and let $H \subset G$ be its infinite proper subgroup. Consider $M = G \setminus H$. We have $M = -M$, and M is syndetic ($I_M = 2$), but M is not vast: $Q - Q \cap M = \varnothing$ for any $Q \subset H$.

However, the "quotient sets" of syndetic sets are vast.

Proposition 1.6. *If a subset S of a group G is syndetic, then $S^{-1}S \in \Phi(G)$ and $J_{S^{-1}S} \le I_S + 1$.*

Proof. Let $T \subset G$ be a finite set for which $G = TS$ and $|T| = I_S$. Take any $P \subset G$ with $|P| \ge |T| + 1$. There exists a $t \in T$ for which $|P \cap tS| > 1$. Given any different $a, b \in P$ such that $a, b \in tS$, we have $b^{-1}a, a^{-1}b \in S^{-1}S$ and $Q^{-1}Q \subset S^{-1}S$ for $Q = \{a, b\} \in [P]^2$. □

This proposition implies the following two assertions.

Corollary 1.7. *Any subgroup of finite index in a group G is vast in G.*

Corollary 1.8. *Any neighborhood U of the identity element in a totally bounded topological group G is vast in G.*

Proof. Let V be a neighborhood of the identity for which $V^{-1}V \subset U$. Since V is syndetic, it follows by Proposition 1.6 that U is vast. □

There are vast sets different from those provided by Proposition 1.6 and Corollaries 1.7 and 1.8. A whole lot of them can be obtained by using the following proposition.

Proposition 1.9. *Let G be a group. If $W \subset G$ and $W \cap W^{-1}W = \varnothing$, then $G \setminus W \in \Phi(G)$ and $J_{G\setminus W} \le 4$.*

Proof. We set $M = G \setminus W$. Take $P \subset G$ with $|P| = 4$; suppose that $P = \{p_0, p_1, p_2, p_3\}$. Let us show that $P^{-1}P \cap (M \cap M^{-1}) \not\subset \{e\}$. Assume that, on the contrary, $P^{-1}P \subset (G \setminus (M \cap M^{-1})) \cup \{e\} = W \cup W^{-1} \cup \{e\}$. Fix any $i \le 4$. For each $j \ne i$, $j \le 4$, we have either $p_i^{-1}p_j \in W$ or $(p_i^{-1}p_j)^{-1} = p_j^{-1}p_i \in W$. Hence the numbers $s_i = |\{j : p_i^{-1}p_j \in W\}|$ and $m_i = |\{j : p_j^{-1}p_i \in W\}|$ satisfy the condition $s_i + m_i \ge 3$. Clearly, $\sum_{i\le 4} s_i = \sum_{i\le 4} m_i$. Therefore, $s_n \ge 2$ for some n. Let i and j be different numbers for which $g = p_n^{-1}p_i \in W$ and $h = p_n^{-1}p_j \in W$. Then either $g^{-1}h \in W$ or $h^{-1}g \in W$. This contradicts the assumption $W \cap W^{-1}W = \varnothing$. Hence $P^{-1}P \cap (M \cap M^{-1}) \ne \{e\}$, i.e., there exist $a, b \in P$ such that $a \ne b$ and $a^{-1}b \in M \cap M^{-1}$. Clearly, for $Q = \{a, b\} \in [P]^2$, we have $Q^{-1}Q \subset M$. □

Unlike syndetic sets, vast sets in a group form a filter by virtue of the following proposition.

Proposition 1.10. *Suppose that G is a group and $M_1, M_2 \in \Phi(G)$. Then $M_1 \cap M_2 \in \Phi(G)$.*

Proof. We can assume without loss of generality that $M_1^{-1} = M_1$ and $M_2^{-1} = M_2$. Proposition 1.2 implies the existence of a positive integer m such that, for any $P \subset G$ with $|P| \ge m$, there exists an $R \subset P$ with $|R| = J_{M_1}$ for which $R^{-1}R \subset M_2$. Since $|R| \ge J_{M_1}$, it follows that $Q^{-1}Q \subset M_1$ for some $Q \in [R]^2$. Hence $Q^{-1}Q \subset M_1 \cap M_2$. □

Propositions 1.3 and 1.10, together with the characterization of a nonrapid filter given in the introduction, imply the following technical statement, which is our main tool for constructing nonclosed discrete sets in groups.

Statement 1.11. *Suppose that G is a countable group with identity element e, X is a set, $f\colon G\to X$ is a finite-to-one map, $f(G)=X$, $\mathscr{F}$ is a free filter on G, and $\mathscr{G}=\{f(F):F\in\mathscr{F}\}$ is a nonrapid free filter on X. Let $(M_n)_{n\in\omega}$ be a sequence of vast subsets of G. Then there exists a sequence $\xi=(x_n)_{n\in\omega}\subset G\setminus\{e\}$ such that*

(i) *$\xi\setminus M_n$ is finite for each $n\in\omega$;*
(ii) *each $F\in\mathscr{F}$ contains g and h such that $f(g)\neq f(h)$ and $g^{-1}h\in\xi$.*

Proof. In view of Propositions 1.3 and 1.10, we can assume without loss of generality that $M_{n+1}\subset M_n$ and $M_n=M_n^{-1}$ for all $n\in\omega$. Since the filter $\mathscr{G}$ is nonrapid, there exists a sequence $(T_n)_{n\in\omega}$ of finite subsets of X such that, given any $F\in\mathscr{F}$, we have $|f(F)\cap T_n|\geq J_{M_n}$ for some $n\in\omega$. We set

$$S_n=\{g^{-1}h: g,h\in f^{-1}(T_n),\ f(g)\neq f(h),\ g^{-1}h\in M_n\}$$

and $\xi=\bigcup_n S_n$.

Let us check that (i) holds. Since the sets S_k are finite, $S_k\subset M_k$, and $M_{k+1}\subset M_k$ for all $k\in\omega$, it follows that $\xi\setminus M_n\subset\bigcup_{k<n}S_k$ is finite for each n.

Let us verify (ii). Take $F\in\mathscr{F}$. We have $|f(F)\cap T_n|\geq J_{M_n}$ for some $n\in\omega$. Choose $P\subset F$ so that $f(P)\subset T_n$, $|P|\geq J_{M_n}$, and $f(g)\neq f(h)$ for any different $g,h\in P$. There exists a $Q=\{g,h\}\in[P]^2$ such that $Q^{-1}Q\subset M_n$. We have $g,h\in F$, $f(g)\neq f(h)$, $g,h\in f^{-1}(T_n)$, and $g^{-1}h\in M_n$. Therefore, $g^{-1}h\in S_n\subset\xi$. □

2. Discrete sequences in topological groups

In the context of topological groups, Statement 1.11 can be refined as follows.

Statement 2.1. *Let G be a countable topological group with identity element e. Suppose that X is a set, $f\colon G\to X$ is a finite-to-one map, $f(G)=X$, $\mathscr{F}$ is a free filter on G converging to e, and $\mathscr{G}=\{f(F):F\in\mathscr{F}\}$ is a nonrapid free filter on X. Suppose also that $(U_n)_{n\in\omega}$ is a decreasing sequence of neighborhoods of e such that $U_n=U_n^{-1}$, $U_{n+1}^3\subset U_n$, and $\bigcap_n U_n=\{e\}$. Finally, let $(H_n)_{n\in\omega}$ be a sequence of subgroups of finite index in G. Then there exists a sequence $\xi=(x_n)_{n\in\omega}\subset G\setminus\{e\}$ such that*

(i) *ξ is discrete and e is its only limit point;*
(ii) *each $F\in\mathscr{F}$ contains g and h such that $f(g)\neq f(h)$ and $g^{-1}h\in\xi$;*
(iii) *$\xi\cap gU_{n+1}$ is finite for any $n\in\omega$ and any $g\in G\setminus U_n$;*
(iv) *$\xi\setminus H_n$ is finite for each $n\in\omega$.*

If, in addition, U_n is syndetic for each $n\in\omega$, then

(v) *$\xi\setminus U_n$ is finite for each $n\in\omega$.*

Proof. Consider $\gamma=\{gU_{n+1}:n\in\omega,\ g\in G\setminus U_n\}$. Let us enumerate the elements of γ: $\gamma=\{W_n\subset G:n\in\omega\}$. For each n, we have $W_n=gU_{k+1}$ for some $k\in\omega$ and $g\in G\setminus U_k$. Hence $W_n\cap W_n^{-1}W_n=gU_{k+1}\cap U_{k+1}^{-1}U_{k+1}=\varnothing$, because $U_{k+1}=U_{k+1}^{-1}$ and $g\notin U_{k+1}^3\subset U_k$. Therefore, by Proposition 1.9, all sets W_n are vast, and by Proposition 1.10 and Corollary 1.7, all intersections $W_n\cap H_n$ are vast as well. Statement 1.11 implies the existence of a sequence $\xi=(x_n)_{n\in\omega}\subset G\setminus\{e\}$ satisfying conditions (ii), (iii), and (iv); (i) follows from (ii) and (iii).

Let us check (v). Take $n\in\omega$. Since $U_{n+2}^{-1}U_{n+2}\subset U_{n+1}$ and the set U_{n+2} is syndetic, it follows from Proposition 1.6 that U_{n+1} is vast. Proposition 1.5 implies the existence of a finite set $Q\subset G\setminus U_n$ for which $G\setminus U_n\subset QU_{n+1}$. According to (iii), $\xi\cap qU_{n+1}$ is finite for each $q\in Q$. Therefore, $\xi\setminus U_n$ is finite. □

Note that in this statement, as well as in Theorem 2.4 and Corollary 2.5 below, the subgroups H_n are not required to be proper or different.

Any countable topological group contains a sequence $(U_n)_{n\in\omega}$ of neighborhoods of the identity element satisfying the assumptions of Statement 2.1. Thus, Statement 2.1 has the following corollary.

Corollary 2.2. *Suppose that G is a countable topological group with identity element e, X is a set, $f\colon G\to X$ is a finite-to-one map, $f(G)=X$, $\mathscr{F}$ is a free filter on G converging to e, and $\mathscr{G}=\{f(F): F\in\mathscr{F}\}$ is a nonrapid free filter on X. Then there exists a sequence $\xi=(x_n)_{n\in\omega}\subset G\setminus\{e\}$ such that*

(i) *ξ is discrete, and e is its only limit point;*
(ii) *each $F\in\mathscr{F}$ contains g and h such that $f(g)\neq f(h)$ and $g^{-1}h\in\xi$.*

Corollary 2.2 implies the following technical assertion needed in what follows.

Statement 2.3. *Suppose that there are no rapid ultrafilters. Let G be a countable topological group with identity element e. Suppose that $Y\subset G$, $e\in\overline{Y}\setminus Y$, and $\{Y_n : n\in\omega\}$ is a partition of Y into finite subsets. Then there exists a sequence $\xi=(x_n)_{n\in\omega}\subset G\setminus\{e\}$ such that*

(i) *ξ is discrete, and e is its only limit point;*
(ii) *$\xi\subset\bigcup_{i\neq j}Y_i^{-1}Y_j$.*

Proof. Take a partition $\mathcal{X}$ of G such that $\{Y_n : n\in\omega\}\subset\mathcal{X}$ and $\{g\}\in\mathcal{X}$ for all $g\in G\setminus Y$. We define $f\colon G\to\mathcal{X}$ to be the natural map taking each element $g\in G$ to the (uniquely determined) element $f(g)$ of $\mathcal{X}$ containing g. Let $\mathscr{F}$ be a free filter on G converging to e and containing Y. Then, by virtue of Corollary 2.2, there exists a sequence $\xi'=(x'_n)_{n\in\omega}\subset G\setminus\{e\}$ satisfying the following conditions:

(i) ξ' is discrete, and e is its only limit point;
(ii) each $F\in\mathscr{F}$ contains g and h such that $f(g)\neq f(h)$ and $g^{-1}h\in\xi'$.

We set $\xi=\xi'\cap Z$, where $Z=\bigcup_{i\neq j}Y_i^{-1}Y_j$. Let us check that $e\in\overline{\xi}$. Take neighborhoods U and V of e in G for which $V^{-1}V\subset U$ and let $F=V\cap Y$; then $F\in\mathscr{F}$. There exist $g,h\in F$ for which $f(g)\neq f(h)$ and $g^{-1}h\in\xi'$, and there exist different $i,j\in\omega$ for which $Y_i=f(g)$ and $Y_j=f(h)$. We have $g^{-1}h\in\xi'\cap Y_i^{-1}Y_j\cap V^{-1}V\subset\xi\cap U$, i.e., $\xi\cap U\neq\varnothing$. $\square$

Now, we can prove our first theorem, which strengthens Theorem 2.1 of [10].

Theorem 2.4. *Let (G,τ) be a countable topological group with identity element e and topology τ, and let $\mathscr{F}$ be a nonrapid free filter on G converging to e. Suppose that $\tau_{\mathrm{m}}\subset\tau$ is a metrizable group topology on G coarser than τ. Finally, suppose that $(H_n)_{n\in\omega}$ is a sequence of subgroups of finite index in G. Then there exists a sequence $\xi=(x_n)_{n\in\omega}\subset G\setminus\{e\}$ such that*

(i) *ξ is discrete, and e is its only limit point both in (G,τ) and in (G,τ_{m});*
(ii) *$\xi\cap F^{-1}F\neq\varnothing$ for any $F\in\mathscr{F}$;*
(iii) *$\xi\setminus H_n$ is finite for each $n\in\omega$.*

If, in addition, (G,τ_{m}) is totally bounded, then

(iv) *ξ converges to e in (G,τ_{m}).*

Proof. Take a sequence $(U_n)_{n\in\omega}$ of neighborhoods of e open in (G,τ_{m}) and such that $(U_n)_n$ is a base of neighborhoods of e in (G,τ_{m}), $U_n=U_n^{-1}$, and $U_{n+1}^3\subset U_n$ for $n\in\omega$. Let $X=G$, and let $f\colon G\to X$ be the identity map. By virtue of

Statement 2.1, there is a sequence $\xi = (x_n)_{n\in\omega} \subset G \setminus \{e\}$ satisfying conditions (i)–(iv) of Statement 2.1. Clearly, this sequence satisfies also conditions (i), (ii), and (iii) of the theorem being proved.

Let us check (iv). Take $n \in \omega$. By Corollary 1.8 the neighborhood U_{n+1} is vast. Proposition 1.5 implies the existence of a finite set $Q \subset G \setminus U_n$ for which $G \setminus U_n \subset QU_{n+1}$. According to Statement 2.1 (iii), $\xi \cap qU_{n+1}$ is finite for each $q \in Q$. Therefore, $\xi \setminus U_n$ is finite. □

Obviously, the topology of any countable topological group can be weakened to a metrizable group topology (see, e.g., [2]). Thus, we obtain the following corollary of Theorem 2.4.

Corollary 2.5. *Let (G, τ) be a countable nondiscrete topological group with identity element e such that the filter of neighborhoods of e is nonrapid. Suppose that $(H_n)_{n\in\omega}$ is a sequence of subgroups of finite index in G. Then there exists a sequence $\xi = (x_n)_{n\in\omega} \subset G \setminus \{e\}$ such that*

(i) *ξ is discrete and e is its only limit point;*
(ii) *$\xi \setminus H_n$ is finite for each $n \in \omega$.*

A special case of this corollary is the following theorem, which is one of the main results of this paper.

Theorem 2.6. *Any countable nondiscrete topological group whose identity element has nonrapid filter of neighborhoods contains a discrete sequence with precisely one limit point.*

The following theorem says that not only does any countable group with nonrapid neighborhood filter of the identity contain a discrete set with one limit point, but it must also contain two such disjoint sets with the same limit point under certain set-theoretic assumptions.

Theorem 2.7. *Let (G, τ) be a countable nondiscrete topological group with identity element e such that the filter of neighborhoods of e is nonrapid, and let $(U_n)_{n\in\omega}$ be a decreasing sequence of neighborhoods of e such that $U_0 = G$, $U_n = U_n^{-1}$, $U_{n+1}^3 \subset U_n$, and $\bigcap_n U_n = \{e\}$. Consider the map $\theta\colon G \setminus \{e\} \to \omega$ defined by $\theta^{-1}(n) = U_n \setminus U_{n+1}$ for each $n \in \omega$. Suppose that there exist no two disjoint discrete sequences $\xi, \xi' \subset G \setminus \{e\}$ each of which has the unique limit point e. Then*

(i) *$\mathrm{Ult}(\theta)(\mathrm{Ult}_e^*(G))$ contains a P-point ultrafilter $\mathscr{U}$.*

If, in addition, U_n is syndetic for each $n \in \omega$, then

(ii) *$\mathscr{U}$ can be mapped to a selective ultrafilter.*

Proof. We set $\mathscr{F}$ to be the filter of neighborhoods of e and f to be the identity map $G \to G$ and apply Statement 2.1. Let $\xi = (x_n)_{n\in\omega} \subset G \setminus \{e\}$ be a sequence with the properties specified in Statement 2.1. For each $n \in \omega$, there exists a $k'_n \in \omega$ such that $\xi \cap x_n U_{k'_n} = \{x_n\}$, because $\xi \cap x_n U_{\theta(x_n)+1}$ is finite and $\bigcap_m U_m = \{e\}$. Let $(k_n)_{n\in\omega} \subset \omega$ be an increasing sequence such that $k_n > k'_n$ and $k_n > \theta(x_n)$. Then (a) the sets $x_n U_{k_n}$ are disjoint and (b) $\overline{\bigcup_n x_n U_{k_n}} \setminus \left(\bigcup_n \overline{x_n U_{k_n}}\right) = \{e\}$. Indeed, if $x_l U_{k_l} \cap x_m U_{k_m} \neq \varnothing$ and, say, $l < m$, then $x_m \in x_l U_{k_l} U_{k_m}^{-1} \subset x_l U_{k_l}^2 \subset x_l U_{k'_l}$, which contradicts the definition of k'_l and, thereby, proves (a). To prove (b), we take any $g \neq e$ and find n for which $g \notin U_n$. By condition (iii) in Statement 2.1, $\xi \cap gU_{n+1}$ is finite, and hence so is the set M of numbers m for which $x_m U_{n+2} \cap gU_{n+2} \neq \varnothing$;

therefore, the intersection $x_lU_{k_l}\cap gU_{n+2}$ can be nonempty only if $l \in M$ or $k_l < n+2$, and the number of such l's is finite.

Let us prove (i). Take an ultrafilter $\mathscr{V} \in \mathrm{Ult}^*_e(G)$ containing ξ. We claim that $\mathscr{U} = \mathrm{Ult}(\theta)(\mathscr{V})$ is P-point. Suppose that, on the contrary, there exists an increasing sequence $(A_n)_{n\in\omega}$ of sets $A_n \subset \omega$ not belonging to $\mathscr{U}$ and such that each $P \in \mathscr{U}$ has infinite intersection with some A_n. We set $B_n = \theta^{-1}(A_n)\cap U_{k_n}\cap\xi$ for $n \in \omega$ and define ξ' as $\bigcup_n x_nB_n$. For each n, $\xi \setminus \theta^{-1}(A_n) \in \mathscr{V}$ and hence $e \notin \overline{\theta^{-1}(A_n)\cap\xi}$: otherwise, we would have two disjoint discrete sequences each of which has the unique limit point e. Therefore, each B_n is a closed discrete set; by virtue of assertions (a) and (b) at the end of the preceding paragraph, the whole sequence ξ' is discrete and cannot have limit points different from e. Note that $\xi\cap\xi' = \varnothing$. Indeed, for each n, $e \notin B_n$ and hence $x_n \notin x_nB_n$; on the other hand, $x_nB_n\cap\xi \subset x_nU_{k_n}\cap\xi = \{x_n\}$. Since $e \in \overline{\xi}$, it follows that $e \notin \overline{\xi'}$, i.e., ξ' is a closed discrete subset of G. Let U be a neighborhood of e with the properties $U = U^{-1}$ and $U^2\cap\xi' = \varnothing$, and let $P = \theta(U\cap\xi)$. We have $P \in \mathscr{U}$; hence there exists an $n \in \omega$ for which $|P \cap A_n| = \aleph_0$. Thus, we can choose $l, m \in \omega$ so that $x_l, x_m \in U\cap\xi$, $\theta(x_l), \theta(x_m) \in A_n$, and $m > k_l$. We have $x_m \in B_l$ and $x_lx_m \in \xi'\cap U^2$. This contradiction proves that $\mathscr{U}$ is a P-point ultrafilter.

To prove the second assertion of the theorem, we need the following lemma, which is also used in the next section.

Lemma 2.8. *Let $\mathscr{U}$ be a free ultrafilter on ω, and let $\phi\colon \omega \to \omega$ be a monotone function such that $\phi(n) > n$ for all $n \in \omega$. Then there exist monotone sequences $(a_n)_{n\in\omega}, (b_n)_{n\in\omega} \subset \omega$ such that $a_n < b_n < \phi(b_n) < a_{n+1}$ for all $n \in \omega$ and $\bigcup_n[a_n, b_n] \in \mathscr{U}$.*

Proof. Let $(c_n)_{n\in\omega} \subset \omega$ be a sequence satisfying the conditions $c_0 = 0$ and $c_{n+1} > \phi(c_n)$. We set $A = \bigcup_n[c_{2n}, c_{2n+1}]$ and $B = \bigcup_n[c_{2n+1}, c_{2n+2}]$. We have $A\cup B = \omega$, so that either $A \in \mathscr{U}$ or $B \in \mathscr{U}$. It remains to set $a_n = c_{2n}$ and $b_n = c_{2n+1}$ in the former case and $a_n = c_{2n+1}$ and $b_n = c_{2n+2}$ in the latter. $\square$

We proceed to prove assertion (ii). Suppose that all U_n are syndetic. Let us show that $\mathscr{U}$ can be mapped to a selective ultrafilter in this case. We can assume without loss of generality that $\theta(x_0) = 0$. Recall that the sequence ξ was chosen to satisfy all conditions in Statement 2.1. By condition (v), $\theta^{-1}(n)\cap\xi$ is finite for each $n \in \omega$. Consider the function $\phi\colon \omega \to \omega$ defined by

$$\phi(n) = \max\{k_m : m \in \omega,\ \theta(x_m) \le n\}$$

for each $n \in \omega$.

By virtue of Lemma 2.8, there exist monotone sequences $(a_n)_{n\in\omega}, (b_n)_{n\in\omega} \subset \omega$ such that $a_n < b_n < \phi(b_n) < a_{n+1}$ for all $n \in \omega$ and $C = \bigcup_n[a_n, b_n] \in \mathscr{U}$. Consider the map $\eta\colon C \to \omega$ defined by $\eta^{-1}(n) = [a_n, b_n]$ for each $n \in \omega$. We set $\mathscr{W} = \mathrm{Ult}(\eta)(\mathscr{U})$ and claim that $\mathscr{W}$ is a Q-point ultrafilter.

Indeed, suppose that, on the contrary, ω can be partitioned into disjoint finite sets A_n, $n \in \omega$, so that, for each $R \in \mathscr{W}$, there exists an $n \in \omega$ such that $|R\cap A_n| > 1$. Let $D = \{n \in \omega : \theta(x_n) \in C\}$. Then the sequence $\xi_D = (x_n)_{n\in D}$ accumulates at e, because $\xi_D = \theta^{-1}(C)\cap\xi \in \mathscr{V}$. For each $n \in D$, we find $\alpha_n \in \omega$ for which $\eta(\theta(x_n)) \in A_{\alpha_n}$ and set

$$B_n = \{x_m \in \xi_D : \theta(x_m) \ge k_n,\ \eta(\theta(x_m)) \in A_{\alpha_n}\}.$$

Let $\xi' = \bigcup_{n\in D} x_n B_n$. Note that each B_n is finite (because A_{α_n} is finite, the map η is finite-to-one by definition, and $\theta \restriction \xi$ is finite-to-one by condition (v) in Statement 2.1), and $B_n \subset U_{k_n}$ (by the definition of the map θ). Thus, for the same reasons as in the proof of assertion (i), ξ' is a discrete sequence having no limit points in $G \setminus \{e\}$, and $\xi' \cap \xi = \varnothing$. By the assumption concerning disjoint sequences with limit point e, we have $e \notin \overline{\xi'}$. Let U be a neighborhood of e such that $U = U^{-1}$ and $U^2 \cap \xi' = \varnothing$. Consider $P = \theta(U \cap \xi_D)$ and $R = \eta(P)$. We have $P \in \mathscr{U}$; therefore, $R \in \mathscr{W}$. By assumption we can find $n \in \omega$ for which $|R \cap A_n| > 1$. Take $r, s \in R \cap A_n$, $r < s$. We have $r = \eta(\theta(x_l))$ and $s = \eta(\theta(x_m))$ for some different $x_l, x_m \in U \cap \xi_D$. This means that $\theta(x_l) \in [a_r, b_r]$ and $\theta(x_m) \in [a_s, b_s]$. By the definition of the sequences (a_n) and (b_n), we have $\theta(x_m) > \phi(b_r)$. On the other hand, since $\theta(x_l) \le b_r$, it follows that $\phi(b_r) \ge k_l$. Therefore, $\theta(x_m) \ge k_l$. Finally, we have $\alpha_l = n$, because $\eta(\theta(x_l)) \in A_n$. Thus, $x_m \in B_l$, whence $x_l x_m \in U^2 \cap \xi'$. This contradiction proves that the ultrafilter $\mathscr{W}$ is Q-point.

To complete the proof of the theorem, it remains to note that the property of being P-point is, obviously, preserved by maps of ultrafilters and that the selective ultrafilters are precisely those which are simultaneously P-points and Q-points. $\square$

Corollary 2.9. *Let (G, τ) be a countable nondiscrete extremally disconnected topological group with identity element e such that the filter of neighborhoods of e is nonrapid. Suppose that $(U_n)_{n\in\omega}$ is a decreasing sequence of clopen neighborhoods of e such that $U_n = U_n^{-1}$, $U_{n+1}^3 \subset U_n$, and $\bigcap_n U_n = \{e\}$. Then the family*

$$\mathscr{U} = \{\{n : V \cap U_n \setminus U_{n+1} \ne \varnothing\} : V \text{ is a neighborhood of } e\}$$

is a P-point ultrafilter on ω. If, moreover, all sets U_n are syndetic, then $\mathscr{U}$ can be mapped to a selective ultrafilter.

Indeed, given any set $S \subset \omega$, we have either $S \in \mathscr{U}$ or $\omega \setminus S \in \mathscr{U}$ by virtue of extremal disconnectedness. Thus, $\mathscr{U}$ is an ultrafilter, and $\{\mathscr{U}\} = \mathrm{Ult}(\theta)(\mathrm{Ult}^*_e(G))$. It remains to apply Theorem 2.7.

3. Discrete sequences in Boolean groups

All countable Boolean groups are isomorphic to each other and to the group $[\omega]^{<\omega}$ of finite subsets of ω with the operation $\triangle$ of symmetric difference defined by $A\triangle B = (A\setminus B)\cup(B\setminus A)$ for $A, B \in [\omega]^{<\omega}$; the zero of $[\omega]^{<\omega}$ is the empty set $\varnothing$. We also use the additive notation: $A + B = A\triangle B$ and $\mathbf{0} = \varnothing$. Given a nonempty set $A \in [\omega]^{<\omega}$, by $\min A$ and $\max A$ we denote the minimum and maximum elements of A as a subset of ω.

In this section, we identify all countable Boolean groups with $[\omega]^{<\omega}$.

The proof of our main theorem on Boolean groups is based on two lemmas.

Lemma 3.1. *Suppose that $\mathscr{U}$ is a free ultrafilter on $[\omega]^{<\omega}$, $\xi = \{X_n : n \in \omega\} \in \mathscr{U}$, and $\lim_{n\to\infty} \min X_n = \infty$. Then there exists a sequence $(\mathcal{Y}_n)_{n\in\omega}$ of finite subsets of ξ such that $\bigcup_{n\in\omega} \mathcal{Y}_n \in \mathscr{U}$ and $\left(\bigcup \mathcal{Y}_i\right) \cap \left(\bigcup \mathcal{Y}_j\right) = \varnothing$ for any different $i, j \in \omega$.*

Proof. Let $\mathscr{V} = \min \mathscr{U} = \{\{\min X : X \in \mathcal{M}\} : \mathcal{M} \in \mathscr{U}\}$. We assume that $\min X_0 = 0$. Given $n \in \omega$, we set $h(n) = \max\{\max X : X \in \xi,\ \min X \le n\}$ and $f(n) = 1 + \max\{h(n), n\}$. Using Lemma 2.8, we choose monotone sequences $(a_n)_{n\in\omega}, (b_n)_{n\in\omega} \subset \omega$ so that $a_n < b_n < f(b_n) < a_{n+1}$ for all $n \in \omega$ and $\bigcup_{n\in\omega}[a_n, b_n] \in \mathscr{V}$. Let $\mathcal{Y}_n = \{X \in \xi : \min X \in [a_n, b_n]\}$. Then $\bigcup_{n\in\omega} \mathcal{Y}_n \in \mathscr{U}$.

Since $\bigcup \mathcal{Y}_n \subset [a_n, a_{n+1}-1]$ for each $n \in \omega$, it follows that the family $\{\bigcup \mathcal{Y}_n : n \in \omega\}$ is disjoint. □

Lemma 3.2. *Let $G = [\omega]^{<\omega}$ be a countable nondiscrete Boolean topological group in which the filter of neighborhoods of zero is nonrapid. Then there exists a sequence $\xi = (X_n)_{n\in\omega} \subset G \setminus \{\mathbf{0}\}$ such that*

(i) *ξ is discrete, and its only limit point is $\mathbf{0}$;*
(ii) *ξ can be partitioned into finite subsets $\mathcal{Y}_n$, $n \in \omega$, so that $(\mathcal{Y}_i + \mathcal{Y}_j) \cap \xi = \varnothing$ for different $i, j \in \omega$.*

Proof. Consider the sets $H_n = [\{m \in \omega : m \geq n\}]^{<\omega}$, $n \in \omega$; these are subgroups of finite index in G. By Corollary 2.5, there exists a discrete sequence $\xi' = (X'_n)_{n\in\omega} \subset G \setminus \{\mathbf{0}\}$ such that $\mathbf{0}$ is its only limit point and $\xi' \setminus H_n$ is finite for each $n \in \omega$. We have $\lim_{n\to\infty} \min X'_n = \infty$. Let $\mathscr{U}$ be an ultrafilter on G converging to $\mathbf{0}$ and containing ξ' as an element. Using Lemma 3.1, we choose a sequence $(\mathcal{Y}_n)_{n\in\omega}$ of finite subsets of ξ' so that $\bigcup_{n\in\omega} \mathcal{Y}_n \in \mathscr{U}$ and $\left(\bigcup \mathcal{Y}_i\right) \cap \left(\bigcup \mathcal{Y}_j\right) = \varnothing$ for any different $i, j \in \omega$. It remains to set $\xi = \bigcup_{n\in\omega} \mathcal{Y}_n$. □

Theorem 3.3. *Suppose that there are no rapid ultrafilters. Let G be a countable nondiscrete Boolean topological group. Then there exist two disjoint discrete sequences $(X_n)_{n\in\omega}, (Y_n)_{n\in\omega} \subset G \setminus \{\mathbf{0}\}$ for each of which $\mathbf{0}$ is a unique limit point.*

Proof. By Lemma 3.2, there is a sequence $\xi = (X_n)_{n\in\omega} \subset G \setminus \{\mathbf{0}\}$ such that

(i) ξ is discrete, and $\mathbf{0}$ is its only limit point;
(ii) ξ can be partitioned into finite subsets $\mathcal{Y}_n$, $n \in \omega$, so that $(\mathcal{Y}_i + \mathcal{Y}_j) \cap \xi = \varnothing$ for different $i, j \in \omega$.

Using Statement 2.3, we find $\xi' = (Y_n)_n \subset G \setminus \{\mathbf{0}\}$ such that

(iii) ξ' is discrete, and $\mathbf{0}$ is its only limit point;
(iv) $\xi' \subset \bigcup_{i\neq j} (\mathcal{Y}_i + \mathcal{Y}_j)$.

It follows from (ii) and (iv) that $\xi \cap \xi' = \varnothing$. □

4. Answers and questions

Theorem 2.6 solves a problem of Protasov [17]. Namely, the following assertion is valid.

Corollary 4.1. *It is consistent with ZFC that any countable nondiscrete topological group contains a nonclosed discrete subset with only one limit point.*

This assertion gives also a partial answer to Arhangel'skii and Collins' question on the existence in ZFC of a nondiscrete nodec topological group [3, Problem 8.1].

According to Theorem 2.7, the existence of a countable nondiscrete topological group containing no two disjoint discrete sequences for each of which the identity is a unique limit point implies the existence of either a rapid ultrafilter or a P-point ultrafilter. As mentioned in the introduction, it is unknown whether the nonexistence of both rapid and P-point ultrafilters is consistent with ZFC. This gives rise to the following question.

Problem 4.2. Does there exist in ZFC a countable nondiscrete topological group containing no two disjoint discrete sequences which have the same unique limit point?

Note that such a group cannot be Boolean by virtue of Theorem 3.3.

Recall that a nondiscrete topological space is said to be *resolvable* if it can be partitioned into two dense subsets; otherwise, a space is *irresolvable.* A topological space is said to be ω-*resolvable* if it can be represented as a countable disjoint union of dense subsets. Any homogeneous regular space containing a countable discrete nonclosed set is ω-resolvable (see [30, Theorem 3.33]). Therefore, Theorem 2.6 implies the following assertion.

Corollary 4.3. *The neighborhood filter of the identity element of any countable nondiscrete non-ω-resolvable topological group is rapid.*

Recall that a topological group G is said to be *maximal* if G with any stronger (not necessarily group) topology has isolated points. Clearly, any maximal group is irresolvable. Moreover, it is known that any maximal group is locally countable and even contains a countable open Boolean subgroup [13] (see also [30, Theorem 5.7]). Therefore, Corollary 4.3 has the following consequence.

Corollary 4.4. *The neighborhood filter of the identity element of any maximal topological group is rapid.*

The existence of a countable nondiscrete ω-irresolvable topological group implies the existence of a P-point in $\beta\omega \setminus \omega$ (see [30, Theorem 12.13]).

Problem 4.5. Does the existence of a countable nondiscrete maximal (irresolvable, ω-irresolvable) topological group imply the existence of a selective ultrafilter?

As is known, if X and Y are countable separated sets in an extremally disconnected space ("separated" means that $\overline{X} \cap Y = X \cap \overline{Y} = \varnothing$), then $\overline{X} \cap \overline{Y} = \varnothing$ (see, e.g., [6, Proposition 1.9]). Combining this with Theorem 3.3 and recalling that any extremally disconnected group contains an open Boolean subgroup, we arrive at the following conclusion.

Corollary 4.6. *The existence of a countable nondiscrete extremally disconnected group implies the existence of a rapid ultrafilter.*

Corollary 4.6 solves Arhangel'skii's problem mentioned in the introduction for countable groups.

Problem 4.7. Is it true that the neighborhood filter of the identity element of any countable nondiscrete extremally disconnected group is rapid?

All examples of nondiscrete extremally disconnected groups known to the authors are constructed in models with selective ultrafilters. Note that the existence of a countable nondiscrete extremally disconnected group containing a nonclosed discrete subset implies that of a P-ultrafilter [29].

Problem 4.8. Does the existence of a countable nondiscrete extremally disconnected group imply that of

(a) a selective ultrafilter;
(b) a P-point ultrafilter;
(c) a Q-point ultrafilter?

Corollary 4.6 can be refined as follows: *If G is a countable nondiscrete extremally disconnected topological group, then some ultrafilter $\mathscr{U} \in \mathrm{Ult}_e(G)$ can be finite-to-one mapped to a rapid ultrafilter on ω.* This suggests the following more specific formulation of Problem 4.8.

Problem 4.9. Let G be a countable nondiscrete extremally disconnected topological group. Does there exist an ultrafilter $\mathscr{U} \in \mathrm{Ult}_e(G)$ that can be mapped to

(a) a selective ultrafilter;
(b) a P-point ultrafilter;
(c) a Q-point ultrafilter?

Acknowledgment

The authors are very grateful to the referee for useful comments, which helped to considerably improve the exposition.

References

[1] Alexandre Arhangelski, *Groupes topologiques extrémalement discontinus* (French), C. R. Acad. Sci. Paris Sér. A-B **265** (1967), A822–A825. MR0222207
[2] A. V. Arhangel'skiĭ, *Cardinal invariants of topological groups. Embeddings and condensations*, Soviet Math. Dokl. **20** (1979), 783–787.
[3] A. V. Arhangel′skiĭ and P. J. Collins, *On submaximal spaces*, Topology Appl. **64** (1995), no. 3, 219–241. MR1342519
[4] N. Bourbaki, *Éléments de mathématique. Algèbre. Chapitre 9* (French), Springer-Verlag, Berlin, 2007. Reprint of the 1959 original. MR2325344
[5] W. W. Comfort and Jan van Mill, *Groups with only resolvable group topologies*, Proc. Amer. Math. Soc. **120** (1994), no. 3, 687–696. MR1209097
[6] Z. Frolík, *Maps of extremally disconnected spaces, theory of types, and applications*, General topology and its relations to modern analysis and algebra, III (Proc. Conf., Kanpur, 1968), Academia, Prague, 1971, pp. 131–142. MR0295305
[7] Andrew M. Gleason, *Projective topological spaces*, Illinois J. Math. **2** (1958), 482–489. MR0121775
[8] Neil Hindman and Dona Strauss, *Algebra in the Stone-Čech compactification*, De Gruyter Textbook, Walter de Gruyter & Co., Berlin, 2012. Theory and applications; Second revised and extended edition [of MR1642231]. MR2893605
[9] J. L. Kelley, *Banach spaces with the extension property*, Trans. Amer. Math. Soc. **72** (1952), 323–326. MR0045940
[10] Valentin Keyantuo and Yevhen Zelenyuk, *Discrete subsets and convergent sequences in topological groups*, Topology Appl. **191** (2015), 137–142. MR3361061
[11] Alain Louveau, *Sur un article de S. Sirota* (French), Bull. Sci. Math. (2) **96** (1972), 3–7. MR0308326
[12] V. I. Malykhin, *Extremally disconnected and similar groups*, Soviet Math. Dokl. **16** (1975), 21–25.
[13] V. I. Malykhin, *On extremally disconnected topological groups*, Russian Math. Surveys **34** (1979), 67–76.
[14] Arnold W. Miller, *There are no Q-points in Laver's model for the Borel conjecture*, Proc. Amer. Math. Soc. **78** (1980), no. 1, 103–106. MR548093
[15] Gabriel Mokobodzki, *Ultrafiltres rapides sur N. Construction d'une densité relative de deux potentiels comparables* (French), Séminaire de Théorie du Potentiel, dirigé par M. Brelot, G. Choquet et J. Deny: 1967/68, Exp. 12, Secrétariat mathématique, Paris, 1969, pp. 22. MR0259148
[16] V. I. Ponomarev and L. B. Šapiro, *Absolutes of topological spaces and of their continuous mappings* (Russian), Uspehi Mat. Nauk **31** (1976), no. 5(191), 121–136. MR0451216
[17] I. V. Protasov, *Discrete subsets of topological groups* (Russian), Mat. Zametki **55** (1994), no. 1, 150–151; English transl., Math. Notes **55** (1994), no. 1-2, 101–102. MR1275313
[18] I. V. Protasov, *Filters and topologies on semigroups* (Russian, with English and Russian summaries), Mat. Stud. **3** (1994), 15–28, 120. MR1692845
[19] F. P. Ramsey, *On a Problem of Formal Logic*, Proc. London Math. Soc. (2) **30** (1929), no. 4, 264–286. MR1576401

[20] Saharon Shelah, *Proper and improper forcing*, 2nd ed., Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998. MR1623206

[21] Olga V. Sipacheva, *Nonexistence of countable extremally disconnected groups with many open subgroups*, Topology Appl. **179** (2015), 193–199. MR3270939

[22] S. M. Sirota, *A product of topological groups, and extremal disconnectedness* (Russian), Mat. Sb. (N.S.) **79 (121)** (1969), 179–192. MR0242988

[23] M. H. Stone, *Algebraic characterizations of special Boolean rings*, Fund. Math. **29** (1937), 223–302.

[24] Dona Papert Strauss, *Extremally disconnected spaces*, Proc. Amer. Math. Soc. **18** (1967), 305–309. MR0210066

[25] Edward L. Wimmers, *The Shelah P-point independence theorem*, Israel J. Math. **43** (1982), no. 1, 28–48. MR728877

[26] R. Grant Woods, *A survey of absolutes of topological spaces*, Topological structures, II (Proc. Sympos. Topology and Geom., Amsterdam, 1978), Math. Centre Tracts, vol. 116, Math. Centrum, Amsterdam, 1979, pp. 323–362. MR565852

[27] E. G. Zelenyuk, *Topological groups with finite semigroups of ultrafilters* (Russian, with English and Russian summaries), Mat. Stud. **6** (1996), 41–52, 160. MR1691123

[28] E. G. Zelenyuk, *Extremal ultrafilters and topologies on groups* (Russian, with English and Russian summaries), Mat. Stud. **14** (2000), no. 2, 121–140. MR1813995

[29] Yevhen Zelenyuk, *On extremally disconnected topological groups*, Topology Appl. **153** (2006), no. 14, 2382–2385. MR2243718

[30] Yevhen G. Zelenyuk, *Ultrafilters and topologies on groups*, De Gruyter Expositions in Mathematics, vol. 50, Walter de Gruyter GmbH & Co. KG, Berlin, 2011. MR2768144

Department of General Topology and Geometry, Mechanics and Mathematics Faculty, M. V. Lomonosov Moscow State University, Leninskie Gory 1, Moscow, 199991 Russia

Email address: erezn@inbox.ru

Department of General Topology and Geometry, Mechanics and Mathematics Faculty, M. V. Lomonosov Moscow State University, Leninskie Gory 1, Moscow, 199991 Russia

Email address: ovsipa@gmail.com