Lego-like spheres and tori

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Abstract

Given a connected surface $\mathbb{F}^2$ with Euler characteristic χ and three integers $b > a \geq 1 < k$, an $(\{a,b\};k)$-$\mathbb{F}^2$ is a $\mathbb{F}^2$-embedded graph, having vertices of degree only k and only a- and b-gonal faces. The main case are (geometric) fullerenes (5, 6; 3)-$\mathbb{S}^2$.

Call an $(\{a,b\};k)$-map *lego-admissible* if either $\frac{p_b}{p_a}$, or $\frac{p_a}{p_b}$ is integer. Call it *lego-like* if it is either ab^f*-lego map*, or $a^f b$*-lego map*, i.e., the face-set is partitioned into $\min(p_a, p_b)$ isomorphic clusters, *legos*, consisting either one a-gon and $f = \frac{p_b}{p_a}$ b-gons, or, respectively, $f = \frac{p_a}{p_b}$ a-gons and one b-gon; the case $f = 1$ we denote also by ab.

Call a $(\{a,b\};k)$-map *elliptic*, *parabolic* or *hyperbolic* if the *curvature* $\kappa_b = 1 + \frac{b}{k} - \frac{b}{2}$ of b-gons is positive, zero or negative, respectively.

All 13 elliptic $(\{a,b\};k)$-$\mathbb{S}^2$ with $(a,b) \neq (1,2)$ are ab^f.

No $(\{1,3\};6)$-$\mathbb{S}^2$ is lego-admissible. For other 7 families of parabolic $(\{a,b\};k)$-$\mathbb{S}^2$, each lego-admissible sphere with $p_a \leq p_b$ is $a^f b$ and an infinity (by *Goldberg–Coxeter operation*) of ab^f-spheres exist.

The number of hyperbolic ab^f $(\{a,b\};k)$-$\mathbb{S}^2$ with $(a,b) \neq (1,3)$ is finite. Such $a^f b$-spheres with $a \geq 3$ have $(a,k) = (3,4), (3,5), (4,3), (5,3)$ or $(3,3)$; their number is finite for each b, but infinite for each of 5 cases (a,k). Any lego-admissible $(\{a,b\};k)$-$\mathbb{S}^2$ with $p_b = 2 \leq a$ is $a^f b$.

We list, explicitly or by parameters, lego-admissible $(\{a,b\};k)$-maps among: hyperbolic spheres, spheres with $a \in \{1,2\}$, spheres with $p_b \in \{2, \frac{p_a}{2}\}$, Goldberg–Coxeter's spheres and $(\{a,b\};k)$-tori.

We present extensive computer search of lego-like spheres: 7 parabolic (p_b-dependent) families, basic examples of all 5 hyperbolic $a^f b$ (b-dependent) families with $a \geq 3$ and toric azulenoids $(\{5,7\};3)$-$\mathbb{T}^2$.