Restrictions of homeomorphisms on compactifications

Miroslav Hušek

There are several results asserting that a homeomorphism between compactifications of X, Y entails a homeomorphism between the spaces X and Y (for instance, results by E.Čech, J.R. Isbell, S.Mrówka). Classical approach used large cardinalities of closed subsets of remainders. Our more general results use convergence. Uniformities play important role in our approach.

Two cases may appear. The first one concerns the situation when the restriction of homeomorphisms $bX \to bY$ maps X onto Y. The second case concerns a situation when homeomorphisms $bX \to bY$ imply existence of homeomorphisms $X \to Y$ – that reminds a kind of Banach-Stone–like theorems. Indeed, if X, Y are uniform spaces and the lattices $U(X), U(Y)$ of uniformly continuous real-valued maps are isomorphic, then the lattices $U^*(X), U^*(Y)$ are isomorphic and, thus, the Samuel-Smirnov compactifications sX, sY are homeomorphic (see [1]). So, if that implies homeomorphism of X, Y, we have an implication: isomorphism of $U(X), U(Y)$ implies homeomorphism of X, Y, i.e. a Banach-Stone–like theorem. Such approach was studied in [2].

Typical results (1-4 deal with restrictions of homeomorphisms):

1. *Let X, Y be proximally complete in their compactifications bX, bY, resp. If every point of X, Y has a linearly ordered neighborhood base and bX, bY are homeomorphic, then X, Y are homeomorphic.*
2. *Let X, Y be complete uniform spaces having linearly ordered bases. If sX, sY are homeomorphic, then X, Y are uniformly homeomorphic.*
3. *If compactifications bX, bY of Čech-complete, proximal complete and pseudoradial spaces X, Y are homeomorphic then X and Y are homeomorphic.*
4. *Let X, Y be sequential spaces with homeomorphic sX, sY. Then realcompletions of X, Y are homeomorphic.*
5. *Let X, Y be uniform spaces having no uniformly discrete subset of Ulam measurable cardinality. If the lattices $U(X)$ and $U(Y)$ are isomorphic, then X, Y are uniformly homeomorphic if X, Y are complete, proximally fine and locally fine.*

In case the lattice isomorphism preserves constants, then the assertion holds if X, Y are complete, finitely dimensional and RE-spaces.

References

[1] Hušek, M.,*Lattices of uniformly continuous functions determine their sublattices of bounded functions*, Topology Appl. 182 (2015), 71–76.

[2] M.Hušek, A.Pulgarín, *When lattices of uniformly continuous maps on X determine X*, Topology Appl. 194 (2015), 228–240

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